

# Fast convergence of a damped inertial dynamics. Link with Nesterov algorithm.

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# 1A. General presentation: dynamical system

Fast dynamical methods for convex minimization.

$$\min \{ \Phi(x) : x \in \mathcal{H} \}.$$

- $\mathcal{H}$  real Hilbert space;  $\|x\|^2 = \langle x, x \rangle$ ;
- $\Phi : \mathcal{H} \rightarrow \mathbb{R}$  convex, continuously differentiable,  $\operatorname{argmin} \Phi \neq \emptyset$ .

Dissipative inertial system, asymptotic vanishing damping.

$$\ddot{x}(t) + \frac{\alpha}{t} \dot{x}(t) + \nabla \Phi(x(t)) = 0.$$

- (SBC)  $\alpha \geq 3$ :  $\Phi(x(t)) - \min_{\mathcal{H}} \Phi \leq \frac{C}{t^2}$ ;
- (APR)  $\alpha > 3$ :  $x(t) \rightarrow x_{\infty} \in \operatorname{argmin} \Phi$  as  $t \rightarrow +\infty$ .

Time discretization: fast Nesterov type algorithms.

## 1B. General presentation: history

Heavy Ball with Friction,  $\gamma > 0$ .

$$(HBF) \quad \ddot{x}(t) + \gamma \dot{x}(t) + \nabla \Phi(x(t)) = 0.$$

- Polyak (1987), Haraux-Jendoubi (1998), A-Goudou-Redont (2000).
- Alvarez (2000),  $\Phi$  convex,  $x(t) \rightarrow x_\infty \in \operatorname{argmin} \Phi$ .

Asymptotic Vanishing Damping,  $\lim_{t \rightarrow +\infty} a(t) = 0$ .

$$(AVD) \quad \ddot{x}(t) + a(t)\dot{x}(t) + \nabla \Phi(x(t)) = 0.$$

- Cabot-Engler-Gaddat (2009)

$$\int_{t_0}^{+\infty} a(t) dt = +\infty \implies \Phi(x(t)) \rightarrow \min_{\mathcal{H}} \Phi.$$

- Su-Boyd-Candès (2014), A-Peypouquet-Redont (2015)

$$a(t) = \frac{\alpha}{t}, \alpha \geq 3 \implies \Phi(x(t)) - \min_{\mathcal{H}} \Phi \leq Ct^{-2}.$$

- 1 General presentation.
- 2 Fast convergence of the values.
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## 2A. Fast convergence of the values

$$(AVD)_\alpha \quad \ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \nabla\Phi(x(t)) = 0.$$

Theorem 1 (Su-Boyd-Candès, NIPS 2014)

Suppose  $\alpha \geq 3$ ,  $t_0 > 0$ ,  $x : [t_0, +\infty[ \rightarrow \mathcal{H}$  is an orbit of  $(AVD)_\alpha$ . Then,

$$\Phi(x(t)) - \min_{\mathcal{H}} \Phi \leq \frac{C}{t^2}.$$

*Proof:* for  $x^* \in S = \operatorname{argmin}\Phi$ , take as a **Lyapunov function**

$$\mathcal{E}_\alpha(t) := \frac{2}{\alpha-1}t^2(\Phi(x(t)) - \inf_{\mathcal{H}} \Phi) + (\alpha-1)\|x(t) - x^*\|^2 + \frac{t}{\alpha-1}\|\dot{x}(t)\|^2.$$

$$\dot{\mathcal{E}}_\alpha(t) + 2\frac{\alpha-3}{\alpha-1}t(\Phi(x(t)) - \min_{\mathcal{H}} \Phi) \leq 0.$$

$$C = t_0^2(\Phi(x_0) - \min_{\mathcal{H}} \Phi) + (\alpha-1)^2 d^2(x_0, S) + t_0^2 \|\dot{x}_0\|^2.$$

$$\mathcal{E}_\alpha(t) := \frac{2}{\alpha-1} t^2 (\Phi(x(t)) - \inf_{\mathcal{H}} \Phi) + (\alpha-1) \|x(t) - x^*\|^2 + \frac{t}{\alpha-1} \|\dot{x}(t)\|^2.$$

Derivation of  $\mathcal{E}_\alpha(\cdot)$  gives

$$\begin{aligned} \dot{\mathcal{E}}_\alpha(t) &:= \frac{4}{\alpha-1} t (\Phi(x(t)) - \inf_{\mathcal{H}} \Phi) + \frac{2}{\alpha-1} t^2 \langle \nabla \Phi(x(t)), \dot{x}(t) \rangle \\ &\quad + 2(\alpha-1) \langle x(t) - x^* + \frac{t}{\alpha-1} \dot{x}(t), \dot{x}(t) \rangle + \frac{1}{\alpha-1} \dot{x}(t) \cdot \dot{x}(t) + \frac{t}{\alpha-1} \ddot{x}(t) \cdot \dot{x}(t) \\ &= \frac{4}{\alpha-1} t (\Phi(x(t)) - \inf_{\mathcal{H}} \Phi) + \frac{2}{\alpha-1} t^2 \langle \nabla \Phi(x(t)), \dot{x}(t) \rangle \\ &\quad + 2(\alpha-1) \langle x(t) - x^* + \frac{t}{\alpha-1} \dot{x}(t), \frac{t}{\alpha-1} \left( \frac{\alpha}{t} \dot{x}(t) + \ddot{x}(t) \right) \rangle. \end{aligned}$$

Then use (AVD) in this last expression to obtain

$$\dot{\mathcal{E}}_\alpha(t) = \frac{4}{\alpha-1} t(\Phi(x(t)) - \inf_{\mathcal{H}} \Phi) + \frac{2}{\alpha-1} t^2 \langle \nabla \Phi(x(t)), \dot{x}(t) \rangle \quad (1)$$

$$- 2t \langle x(t) - x^* + \frac{t}{\alpha-1} \dot{x}(t), \nabla \Phi(x(t)) \rangle \quad (2)$$

$$= \frac{4}{\alpha-1} t(\Phi(x(t)) - \inf_{\mathcal{H}} \Phi) - 2t \langle x(t) - x^*, \nabla \Phi(x(t)) \rangle. \quad (3)$$

By convexity of  $\Phi$

$$\Phi(x^*) \geq \Phi(x(t)) + \langle x^* - x(t), \nabla \Phi(x(t)) \rangle.$$

Replacing in (3) we obtain

$$\dot{\mathcal{E}}_\alpha(t) + \left(2 - \frac{4}{\alpha-1}\right) t(\Phi(x(t)) - \inf_{\mathcal{H}} \Phi) \leq 0.$$

$$\dot{\mathcal{E}}_\alpha(t) + 2 \frac{\alpha-3}{\alpha-1} t(\Phi(x(t)) - \inf_{\mathcal{H}} \Phi) \leq 0.$$

## 2B. $\mathcal{O}(\frac{1}{t^2})$ as the worst possible case

$\mathcal{H} = \mathbb{R}$ ,  $\Phi(x) = c|x|^\gamma$ ,  $c > 0$ ,  $\gamma > 0$ , parameters.

$$\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + c\gamma x(t)^{\gamma-1} = 0.$$

Nonnegative completely damped solutions of  $(AVD)_\alpha$ :

$$x(t) = \frac{1}{t^\theta}, \quad \theta > 0.$$

Replacing  $x(\cdot)$  in  $(AVD)_\alpha$  gives  $\gamma > 2$ ,  $\theta = \frac{2}{\gamma-2}$ ,  $\alpha > \frac{\gamma}{\gamma-2}$ , and

$$\Phi(x(t)) = \frac{2}{\gamma(\gamma-2)} \left( \alpha - \frac{\gamma}{\gamma-2} \right) \frac{1}{t^{\frac{2\gamma}{\gamma-2}}}.$$

As  $\gamma \uparrow +\infty$ ,  $\frac{2\gamma}{\gamma-2} \downarrow 2$ :  $\Phi$  becomes very **flat** around its minimizer.



## 2C. Strong convexity: faster rate of convergence

Convergence rates increase indefinitely for larger values of  $\alpha$ .

### Theorem 2 (SBC, APR)

Suppose that  $\Phi : \mathcal{H} \rightarrow \mathbb{R}$  is strongly convex. Let  $x(\cdot)$  be an orbit of  $(AVD)_\alpha$ , with  $\alpha > 3$ . Then

- $x(t)$  converges strongly to the unique element  $x^* \in \operatorname{argmin}\Phi$ ;
- $\Phi(x(t)) - \min_{\mathcal{H}} \Phi = \mathcal{O}(t^{-\frac{2}{3}\alpha})$ ;
- $\|\dot{x}(t)\|^2 = \mathcal{O}(t^{-\frac{2}{3}\alpha})$ ;
- $\|x(t) - x^*\|^2 = \mathcal{O}(t^{-\frac{2}{3}\alpha})$ .

*Proof:* use the Lyapunov function  $\mathcal{E}_\lambda^p$  with  $p = \frac{2}{3}(\alpha - 3)$ ,  $\lambda = \frac{2}{3}\alpha$

$$\mathcal{E}_\lambda^p(t) := t^p \left( t^2(\Phi(x(t)) - \min_{\mathcal{H}} \Phi) + \frac{1}{2} \|\lambda(x(t) - x^*) + t\dot{x}(t)\|^2 \right).$$

## 2D. Example $\Phi(x) = \frac{1}{2}\|x\|^2$ . Role of Bessel functions

$$(\text{AVD})_\alpha \quad \ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + x(t) = 0.$$

Solution of  $(\text{AVD})_\alpha$  with Cauchy data  $x(0) = x_0$ ,  $\dot{x}(0) = 0$ :

$$x(t) = 2^{\frac{\alpha-1}{2}} \Gamma\left(\frac{\alpha+1}{2}\right) \frac{J_{\frac{\alpha-1}{2}}(t)}{t^{\frac{\alpha-1}{2}}} x_0.$$

$J_{\frac{\alpha-1}{2}}(\cdot)$ : first kind Bessel function of order  $\frac{\alpha-1}{2}$ . For large  $t$ ,

$$J_\alpha(t) = \sqrt{\frac{2}{\pi t}} \left( \cos\left(t - \frac{\pi\alpha}{2} - \frac{\pi}{4}\right) + \mathcal{O}\left(\frac{1}{t}\right) \right).$$

Hence

$$\Phi(x(t)) - \min_{\mathcal{H}} \Phi = \mathcal{O}(t^{-\alpha}).$$

Compare with  $\mathcal{O}(t^{-\frac{2}{3}\alpha})$ , valid for arbitrary strongly convex functions.

## 2E. Case $\operatorname{argmin} \Phi = \emptyset$ .

### Theorem 3 (APR)

Suppose  $\Phi : \mathcal{H} \rightarrow \mathbb{R}$  convex,  $\operatorname{argmin} \Phi$  possibly empty. Let  $x(\cdot)$  be an orbit of  $(\operatorname{AVD})_\alpha$  with  $\alpha > 1$ . Then,

$$\lim_{t \rightarrow +\infty} \Phi(x(t)) = \inf_{\mathcal{H}} \Phi.$$

Moreover, if  $\inf_{\mathcal{H}} \Phi > -\infty$ , then  $\lim_{t \rightarrow +\infty} \|\dot{x}(t)\| = 0$ .

Fast convergence may not be satisfied in this case:

$$\Phi(x) = \frac{c}{x^\theta}, \text{ with } c = \frac{2(2\alpha + \theta(\alpha - 1))}{\theta(2 + \theta)^2}.$$

Then  $x(t) = t^{\frac{2}{2+\theta}}$  is solution of  $(\operatorname{AVD})_\alpha$ . We have  $\inf_{\mathcal{H}} \Phi = 0$ , and

$$\Phi(x(t)) = \frac{c}{t^{\frac{2\theta}{2+\theta}}}.$$

### 3A. Weak convergence of the orbits

$$(\text{AVD})_\alpha \quad \ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \nabla\Phi(x(t)) = 0.$$

#### Theorem 4 (APR)

Suppose  $\alpha > 3$ . Let  $x : [t_0, +\infty[ \rightarrow \mathcal{H}$  be an orbit of  $(\text{AVD})_\alpha$ . Then,

- $x(t) \rightharpoonup x^* \in \operatorname{argmin}\Phi$  *weakly* as  $t \rightarrow +\infty$ ;
- $\lim_{t \rightarrow +\infty} \|\dot{x}(t)\| = 0$ ,  $\|\dot{x}(t)\| \leq \frac{C}{t}$ ,  $\int_{t_0}^{+\infty} t \|\dot{x}(t)\|^2 dt < +\infty$ ;
- $\Phi(x(t)) - \min_{\mathcal{H}} \Phi \leq \frac{C}{t^2}$ ,  $\int_{t_0}^{+\infty} t \left( \Phi(x(t)) - \min_{\mathcal{H}} \Phi \right) dt < +\infty$ ;
- $\lim_{t \rightarrow +\infty} \frac{1}{t^\alpha} \int_{t_0}^t \tau^\alpha \|\ddot{x}(\tau)\|^2 d\tau = 0$ .

### 3B. Proof of the convergence results

#### Lemma (Opial)

Let  $S \subset \mathcal{H}$ ,  $S \neq \emptyset$ , and  $x : [t_0, +\infty[ \rightarrow \mathcal{H}$  a map. Assume that

- (i) for every  $z \in S$ ,  $\lim_{t \rightarrow +\infty} \|x(t) - z\|$  exists;
- (ii) every weak sequential cluster point of  $x(\cdot)$  belongs to  $S$ .

Then,  $w - \lim_{t \rightarrow +\infty} x(t) = x_\infty$  exists, for some element  $x_\infty \in S$ .

#### Lemma (differential inequality)

Let  $t_0 > 0$ ,  $\alpha > 1$ , and  $w : [t_0, +\infty[ \rightarrow \mathbb{R}$  that satisfies

$$\dot{w}(t) + \frac{\alpha}{t} w(t) \leq g(t),$$

for some  $g : [t_0, +\infty[ \rightarrow \mathbb{R}^+$  such that  $t \mapsto tg(t) \in L^1(t_0, +\infty)$ . Then

$$w^+ \in L^1(t_0, +\infty).$$

### 3C. Proof of the convergence results

- *Step 1.* Given  $x^* \in \operatorname{argmin}\Phi$ , set  $h(t) := \frac{1}{2}\|x(t) - x^*\|^2$ .

$$\dot{h}(t) = \langle x(t) - x^*, \dot{x}(t) \rangle,$$

$$\ddot{h}(t) = \langle x(t) - x^*, \ddot{x}(t) \rangle + \|\dot{x}(t)\|^2.$$

Combining these two equations, and using  $(AVD)_\alpha$ , we obtain

$$\ddot{h}(t) + \frac{\alpha}{t}\dot{h}(t) = \|\dot{x}(t)\|^2 + \langle x(t) - x^*, \ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) \rangle, \quad (4)$$

$$= \|\dot{x}(t)\|^2 + \langle x(t) - x^*, -\nabla\Phi(x(t)) \rangle. \quad (5)$$

By monotonicity of  $\nabla\Phi$  and  $\nabla\Phi(x^*) = 0$ , we infer

$$\ddot{h}(t) + \frac{\alpha}{t}\dot{h}(t) \leq \|\dot{x}(t)\|^2. \quad (6)$$

The next step is to prove that  $\int_{t_0}^{+\infty} t\|\dot{x}(t)\|^2 dt < +\infty$ .

### 3D. Proof of the convergence result

- *Step 2.* From

$$\dot{\mathcal{E}}_\alpha(t) + 2\frac{\alpha-3}{\alpha-1}t(\Phi(x(t)) - \min_{\mathcal{H}}\Phi) \leq 0,$$

and  $\alpha > 3$ , we deduce that

$$\int_{t_0}^{+\infty} t \left( \Phi(x(t)) - \min_{\mathcal{H}}\Phi \right) dt < +\infty. \quad (7)$$

Then, take the scalar product of  $(\text{AVD})_\alpha$  by  $t^2\dot{x}(t)$ , and integrate

$$\frac{1}{2}t^2 \frac{d}{dt} \|\dot{x}(t)\|^2 + \alpha t \|\dot{x}(t)\|^2 + t^2 \frac{d}{dt} \Phi(x(t)) \leq 0.$$

$$\frac{t^2}{2} \|\dot{x}(t)\|^2 + (\alpha-1) \int_{t_0}^t \tau \|\dot{x}(\tau)\|^2 d\tau \leq C + 2 \int_{t_0}^t \tau (\Phi(x(\tau)) - \min_{\mathcal{H}}\Phi) d\tau.$$

$$\int_{t_0}^{+\infty} t \|\dot{x}(t)\|^2 dt < +\infty. \quad (8)$$

### 3E. Proof of the convergence result

- *Step 3.* Asymptotic behaviour of  $\dot{x}(t)$ . From

$$\frac{t^2}{2} \|\dot{x}(t)\|^2 + (\alpha - 1) \int_{t_0}^t \tau \|\dot{x}(\tau)\|^2 d\tau \leq C + 2 \int_{t_0}^t \tau (\Phi(x(\tau)) - \min_{\mathcal{H}} \Phi) d\tau$$

we deduce that  $\|\dot{x}(t)\| \leq \frac{C}{t}$ ,  $\lim_{t \rightarrow +\infty} \|\dot{x}(t)\| = 0$ .

- *Step 4.* Let us show that  $\lim_{t \rightarrow +\infty} \frac{1}{t^\alpha} \int_{t_0}^t \tau^\alpha \|\ddot{x}(\tau)\|^2 d\tau = 0$ .  
Let us return to (4)

$$\ddot{h}(t) + \frac{\alpha}{t} \dot{h}(t) + \langle x(t) - x^*, \nabla \Phi(x(t)) \rangle = \|\dot{x}(t)\|^2.$$

$\nabla \Phi$  is Lipschitz continuous on bounded sets. By Baillon-Haddad theorem, it is  $\frac{1}{L}$ -cocoercive on a ball containing the trajectory:

$$\langle x(t) - x^*, \nabla \Phi(x(t)) - \nabla \Phi(x^*) \rangle \geq \frac{1}{L} \|\nabla \Phi(x(t)) - \nabla \Phi(x^*)\|^2.$$



### 3F. Proof of the convergence result

Combining the two above equations, and using  $\nabla\Phi(x^*) = 0$ , we obtain

$$\ddot{h}(t) + \frac{\alpha}{t}\dot{h}(t) + \frac{1}{L}\|\nabla\Phi(x(t))\|^2 \leq \|\dot{x}(t)\|^2.$$

Replacing  $\nabla\Phi(x(t)) = -\ddot{x}(t) - \frac{\alpha}{t}\dot{x}(t)$

$$\ddot{h}(t) + \frac{\alpha}{t}\dot{h}(t) + \frac{1}{L}\|\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t)\|^2 \leq \|\dot{x}(t)\|^2.$$

Developing

$$\ddot{h}(t) + \frac{\alpha}{t}\dot{h}(t) + \frac{1}{L}\|\ddot{x}(t)\|^2 + \frac{\alpha}{Lt}\frac{d}{dt}\|\dot{x}(t)\|^2 \leq \|\dot{x}(t)\|^2.$$

Then integrate, and apply Fubini's theorem. □

## 4. Strong convergence results

$$(\text{AVD})_\alpha \quad \ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \nabla\Phi(x(t)) = 0.$$

### Theorem 5 (APR)

Suppose  $\alpha > 3$ , and one of the following properties is satisfied by  $\Phi$ :

- $\text{int}(\text{argmin } \Phi) \neq \emptyset$ ;
- $\Phi$  is an even function (i.e.,  $\Phi(-x) = \Phi(x)$ );
- $\Phi$  is uniformly convex.

Then, for any orbit  $x(\cdot)$  of  $(\text{AVD})_\alpha$ , there exists  $x^* \in \text{argmin}\Phi$  such that

$$x(t) \rightarrow x^* \in \text{argmin}\Phi \text{ *strongly* in } \mathcal{H} \text{ as } t \rightarrow +\infty.$$

## 5A. Fast associated algorithms. Link to Nesterov.

Non-smooth structured convex minimization problem:

$$\min \{ \Phi(x) + \Psi(x) : x \in \mathcal{H} \}.$$

- $\Phi : \mathcal{H} \rightarrow \mathbb{R} \cup \{+\infty\}$  closed, convex, proper ;
- $\Psi : \mathcal{H} \rightarrow \mathbb{R}$  convex, differentiable,  $\nabla\Psi$  Lipschitz continuous.

Optimal solutions:

$$\partial\Phi(x) + \nabla\Psi(x) \ni 0.$$

Dynamical approach via the differential inclusion

$$\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \partial\Phi(x(t)) + \nabla\Psi(x(t)) \ni 0.$$

## 5B. Fast associated algorithms. Link to Nesterov.

$$\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \partial\Phi(x(t)) + \nabla\Psi(x(t)) \ni 0.$$

$\Theta := \Phi + \Psi$ ,  $\Theta : \mathcal{H} \rightarrow \mathbb{R} \cup \{+\infty\}$  closed convex, proper.

$$\ddot{x}(t) + a(t)\dot{x}(t) + \partial\Theta(x(t)) \ni 0.$$

- $a(t) \equiv \gamma > 0$ ,  $\dim \mathcal{H} < +\infty$ , Schatzman, A-Cabot-Redont.  
 $x(\cdot)$  loc. Lipschitz;  $\dot{x}(\cdot)$  bounded variation;  $\ddot{x}(\cdot)$  bounded measure.  
Nonuniqueness (shocks).
- $a(t) = \frac{\alpha}{t}$ ,  $\alpha \geq 3$ . Lyapunov analysis is still valid: convex subdifferential inequalities, generalized derivation chain rule.

## 5C. Fast associated algorithms. Link to Nesterov.

$$\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \partial\Phi(x(t)) + \nabla\Psi(x(t)) \ni 0.$$

- **Implicit** discretization / **nonsmooth** function  $\Phi$ .
- **Explicit** discretization / **smooth** function  $\Psi$ .

Time step  $h > 0$ ,  $t_k = kh$ ,  $x_k = x(t_k)$ . Finite difference scheme

$$\frac{1}{h^2}(x_{k+1} - 2x_k + x_{k-1}) + \frac{\alpha}{kh^2}(x_k - x_{k-1}) + \partial\Phi(x_{k+1}) + \nabla\Psi(y_k) \ni 0.$$

$$x_{k+1} + h^2\partial\Phi(x_{k+1}) \ni \left(x_k + \left(1 - \frac{\alpha}{k}\right)(x_k - x_{k-1})\right) - h^2\nabla\Psi(y_k).$$

**Natural choice (Nesterov):**  $y_k = x_k + \left(1 - \frac{\alpha}{k}\right)(x_k - x_{k-1})$ .

## 5D. Fast associated algorithms. Link to Nesterov.

### Proximal mapping, resolvent

$$\text{prox}_{\gamma\Phi}(x) := \operatorname{argmin}_{\xi \in \mathcal{H}} \left\{ \Phi(\xi) + \frac{1}{2\gamma} \|\xi - x\|^2 \right\} = (I + \gamma\partial\Phi)^{-1}(x).$$

$$\begin{cases} y_k = x_k + \left(1 - \frac{\alpha}{k}\right) (x_k - x_{k-1}); \\ x_{k+1} = \text{prox}_{h^2\Phi}(y_k - h^2\nabla\Psi(y_k)). \end{cases} \quad (9)$$

Equivalent formulation  $\left(1 - \frac{\alpha}{k+\alpha-1} = \frac{k-1}{k+\alpha-1}\right)$ :

$$(\text{AVD} - \text{algo})_{\alpha} \begin{cases} y_k = x_k + \frac{k-1}{k+\alpha-1} (x_k - x_{k-1}); \\ x_{k+1} = \text{prox}_{h^2\Phi}(y_k - h^2\nabla\Psi(y_k)). \end{cases} \quad (10)$$

## 5E. Fast associated algorithms. Link to Nesterov.

Set  $s = h^2$ .

$$(\text{AVD - algo})_{\alpha} \begin{cases} y_k = x_k + \frac{k-1}{k+\alpha-1}(x_k - x_{k-1}); \\ x_{k+1} = \text{prox}_{s\Phi}(y_k - s\nabla\Psi(y_k)). \end{cases}$$

- Proximal inertial algo.: A-Alvarez, Moudafi-Oliny, Lorenz-Pock.
- $\alpha = 3$ : Nesterov, Güler, Beck-Teboulle (FISTA)

$$(\text{FISTA}) \begin{cases} y_k = x_k + \frac{k-1}{k+2}(x_k - x_{k-1}); \\ x_{k+1} = \text{prox}_{s\Phi}(y_k - s\nabla\Psi(y_k)). \end{cases}$$

- $\alpha \geq 3$ . Recent studies Chambolle-Dossal, Su-Boyd-Candès, APR.

## 5F. Fast associated algorithms. Link to Nesterov.

$$(\text{AVD} - \text{algo})_\alpha \begin{cases} y_k = x_k + \frac{k-1}{k+\alpha-1}(x_k - x_{k-1}); \\ x_{k+1} = \text{prox}_{s\Phi}(y_k - s\nabla\Psi(y_k)). \end{cases}$$

### Theorem 6 (Chambolle-Dossal, Su-Boyd-Candès)

- $\Phi : \mathcal{H} \rightarrow \mathbb{R} \cup \{+\infty\}$  closed convex proper;
- $\Psi : \mathcal{H} \rightarrow \mathbb{R}$  convex differentiable,  $\nabla\Psi$   $L$ -Lipschitz continuous;
- $S = \text{argmin}(\Phi + \Psi) \neq \emptyset$ ,  $s < \frac{1}{L}$ ,  $\alpha > 3$ .

Let  $(x_k)$  be a sequence generated by  $(\text{AVD} - \text{algo})_\alpha$ . Then,

- $x_k \rightharpoonup x^* \in \text{argmin}(\Phi + \Psi)$  weakly as  $k \rightarrow +\infty$ .
- $(\Phi + \Psi)(x_k) - \min_{\mathcal{H}}(\Phi + \Psi) \leq \frac{C}{k^2}$ .
- $\sum_k k \|x_k - x_{k-1}\|^2 < +\infty$ ,  $\|x_k - x_{k-1}\| \leq \frac{C}{k}$ .



## 5G. Fast associated algorithms. Proof of convergence

*Step one.*  $k \mapsto \mathcal{E}_k$  is the correspondent of  $\mathcal{E}_\alpha(\cdot)$ :

$$\mathcal{E}_k = \frac{2s}{\alpha - 1} (k + \alpha - 2)^2 (\Theta(x_k) - \Theta^*) + \frac{1}{\alpha - 1} \|(k + \alpha - 1)y_k - kx_k - (\alpha - 1)x^*\|^2$$

$\mathcal{E}_k$  is a strict Lyapunov function: for any  $k \in \mathbb{N}$

$$\mathcal{E}_k + \frac{2s}{\alpha - 1} \left( (\alpha - 3)(k + \alpha - 2) + 1 \right) (\Theta(x_k) - \inf \Theta) \leq \mathcal{E}_{k-1}.$$

Fast convergence properties

$$\begin{aligned} (\Phi + \Psi)(x_k) - \min_{\mathcal{H}}(\Phi + \Psi) &\leq \frac{C}{k^2} \\ \sum_k k \left( (\Phi + \Psi)(x_k) - \min_{\mathcal{H}}(\Phi + \Psi) \right) &< +\infty. \end{aligned}$$

## 5H. Fast associated algorithms. Proof of convergence

*Step two.* Next step consists in obtaining the energy estimate

$$\sum_k k \|x_k - x_{k-1}\|^2 < +\infty,$$

= discrete version of the continuous energy estimate

$$\int_{t_0}^{\infty} t \|\dot{x}(t)\|^2 dt < +\infty.$$

*Step three.* The final step is to apply Opial's lemma. Using the previous estimates, it is a direct adaptation of the classical proof of the convergence of proximal-like inertial algorithms. It is a parallel argument to that using the differential inequality with  $\|x_k - x^*\|^2$  instead of  $\|x(t) - x^*\|^2$ , and  $x^* \in \operatorname{argmin}(\Phi + \Psi)$ . □

## 6. Related systems. Case $a(t) = \frac{1}{t^\gamma}$

$$\ddot{x}(t) + \frac{1}{t^\gamma} \dot{x}(t) + \nabla \Phi(x(t)) = 0.$$

Global energy :  $W(t) = \frac{1}{2} \|\dot{x}(t)\|^2 + \Phi(x(t)) - \min \Phi.$

$a(t) = \frac{1}{t^\gamma}$	$\gamma = 0$	$0 < \gamma < 1$	$\gamma = 1, a(t) = \frac{\alpha}{t}, \alpha > 3$
$W(t) \rightarrow 0$	$\mathcal{O}(\frac{1}{t})$	$\circ(\frac{1}{t^{1+\bar{\gamma}}}), \forall \bar{\gamma} < \gamma$	$\mathcal{O}(\frac{1}{t^2})$

- $a(t) = 1, \gamma = 0$  (HBF), Alvarez (2000).
- $a(t) = \frac{1}{t^\gamma}, 0 < \gamma < 1$ , Cabot-Frankel (2012), R. May (2015).
- $a(t) = \frac{\alpha}{t}, \alpha \geq 3$ , SBC (2014), APR (2015).

## 7A. Related dynamics. Hessian driven damping

- $\Phi : \mathcal{H} \rightarrow \mathbb{R}$  convex,  $\mathcal{C}^2$ ,  $\operatorname{argmin}\Phi \neq \emptyset$ ,  $\alpha > 0$ ,  $\beta > 0$ .

$$\text{(DIN-AVD)} \quad \ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \beta\nabla^2\Phi(x(t))\dot{x}(t) + \nabla\Phi(x(t)) = 0.$$

### Theorem 7 (APR)

- $\alpha > 0$  :  $\lim_{t \rightarrow +\infty} \Phi(x(t)) = \min \Phi$ ,  $\lim_{t \rightarrow +\infty} \|\dot{x}(t)\| = 0$ .
- $\alpha \geq 3$  :
  - $\Phi(x(t)) - \min_{\mathcal{H}} \Phi \leq \frac{C}{t^2}$ ;
  - $\int_0^{\infty} t^2 \|\nabla\Phi(x(t))\|^2 dt < +\infty$ ;
  - $\lim_{t \rightarrow +\infty} \|\dot{x}(t)\| = \lim_{t \rightarrow +\infty} \|\ddot{x}(t)\| = \lim_{t \rightarrow +\infty} \|\nabla\Phi(x(t))\| = 0$ .
- $\alpha > 3$  :  $x(t)$  converges weakly to a minimizer of  $\Phi$ .

## 7B. (DIN-AVD) with two potentials

$$\min \{ \phi(x) + \Psi(x) : x \in \mathcal{H} \}, \Psi \text{ smooth, } \phi \text{ nonsmooth.}$$

$$\text{(DIN - AVD)} \quad \begin{cases} \dot{x}(t) + \beta \partial \phi(x(t)) - \left( \frac{1}{\beta} - \frac{\alpha}{t} \right) x(t) - y(t) \ni 0; \\ \dot{y}(t) + \nabla \Psi(x(t)) + \frac{1}{\beta} \left( \frac{1}{\beta} - \frac{\alpha}{t} + \frac{\alpha\beta}{t^2} \right) x(t) + \frac{1}{\beta} y(t) = 0. \end{cases}$$

Equivalent equation,  $\phi = \Phi$  smooth:

$$\ddot{x}(t) + \frac{\alpha}{t} \dot{x}(t) + \beta \nabla^2 \Phi(x(t)) \dot{x}(t) + \nabla \Phi(x(t)) + \nabla \Psi(x(t)) = 0.$$

### Theorem 8 (APR)

Let  $(x(\cdot), y(\cdot))$  be an orbit of (DIN - AVD),  $\alpha > 0$ . Then

$$\lim_{t \rightarrow +\infty} \Theta(x(t)) = \min \Theta, \quad \lim_{t \rightarrow +\infty} \|\dot{x}(t)\| = 0.$$

## 7C. (DIN-AVD) algorithm with two potentials

Time step  $h > 0$ ,  $t_k = kh$ ,  $x_k = x(t_k)$ ,  $y_k = y(t_k)$ ,  $(x_0, y_0) \in \mathcal{H} \times \mathcal{H}$ :

$$\begin{cases} 0 \in \frac{x_{k+1} - x_k}{h} + \beta \partial \phi(x_{k+1}) - \left(\frac{1}{\beta} - \frac{\alpha}{kh}\right)x_k - y_k \\ 0 = \frac{y_{k+1} - y_k}{h} + \nabla \Psi(x_{k+1}) + \frac{1}{\beta} \left(\frac{1}{\beta} - \frac{\alpha}{kh} + \frac{\alpha\beta}{k^2 h^2}\right)x_{k+1} + \frac{1}{\beta} y_{k+1} \end{cases}$$

$$\begin{cases} x_{k+1} = \text{prox}_{\beta h \phi} \left( \left(1 + h \left(\frac{1}{\beta} - \frac{\alpha}{kh}\right)\right) x_k + h y_k \right) \\ y_{k+1} = \frac{\beta}{\beta + h} y_k - \frac{h}{\beta + h} \left(\frac{1}{\beta} - \frac{\alpha}{kh} + \frac{\alpha\beta}{k^2 h^2}\right) x_{k+1} - \frac{h\beta}{\beta + h} \nabla \Psi(x_{k+1}). \end{cases}$$

First-order dynamic  $/ (x_k, y_k)$ :  $(x_k, y_k) \rightarrow (x_{k+1}, y_k) \rightarrow (x_{k+1}, y_{k+1})$ .

## 8. Related dynamics. Adaptive restart (SBC)

**Strategy:** maintain high velocity along the orbit.

$$(AVD)_\alpha \quad \ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \nabla\Phi(x(t)) = 0, \quad x(0) = x_0, \quad \dot{x}(0) = 0.$$

**Restarting time:**  $T(\Phi, x_0) = \sup\{t > 0, \forall \tau \in ]0, t[, \frac{d}{d\tau}\|\dot{x}(\tau)\|^2 > 0\}$ .

Before time  $T(\Phi, x_0)$ ,  $t \mapsto \Phi(x(t))$  decreases:

$$\frac{d}{dt}\Phi(x(t)) = \langle \nabla\Phi(x(t)), \dot{x}(t) \rangle = -\frac{\alpha}{t}\|\dot{x}(t)\|^2 - \frac{1}{2}\frac{d}{dt}\|\dot{x}(t)\|^2 \leq 0.$$

At time  $T(\Phi, x_0)$ , stop and restart, and so on.

### Theorem 9 (SBC), linear convergence

Suppose  $\Phi : \mathcal{H} \rightarrow \mathbb{R}$  strongly convex,  $\nabla\Phi$  Lipschitz continuous,  $\alpha \geq 3$ .  
Let  $x_{sr}(\cdot)$  be an orbit of the speed restarting dynamic. Then

$$\Phi(x_{sr}(t)) - \min_{\mathcal{H}} \Phi \leq c_1 e^{-c_2 t}.$$

# Annex 1. Stochastic gradient descent algorithm (CEG)

$$\forall x \in \mathbb{R}^d \quad \Phi(x) := \int_{\Omega} \phi(x, \omega) \mu(d\omega).$$

$(\omega_n)_{n \geq 1}$  is a sequence of independent identically distributed variables

$$X_{n+1} = X_n - \epsilon_n \nabla \phi(X_n, \omega_{n+1}).$$

The stochastic approximation can be numerically improved:

$$X_{n+1} = X_n - \epsilon_{n+1} \frac{\sum_i^n \epsilon_i \nabla \phi(X_i, \omega_{i+1})}{\sum_i^n \epsilon_i}$$

Limit ODE ( $n \rightarrow +\infty$ ,  $\sum \epsilon_n = +\infty$ ,  $\sum \epsilon_n^p < +\infty$  for some  $p > 1$ )

$$s\ddot{X}(s) + \dot{X}(s) + \nabla \Phi(X(s)) = 0.$$

Time rescaling  $t = 2\sqrt{s}$  gives  $\ddot{X}(t) + \frac{1}{t}\dot{X}(t) + \nabla \Phi(X(t)) = 0.$



## Annex 2. Complexity aspects

In 1983, Nemirovsky & Yudin proved lower bounds on the complexity of first-order methods (number of subgradient calls needed to achieve a given accuracy) for convex optimization under various regularity assumptions for the objective functions. See also Nesterov (2004).





- 1 They constructed convex, piecewise linear functions in dimensions  $n > k$ , where no first-order method can have function values more accurate than  $\mathcal{O}(\frac{1}{\sqrt{k}})$  after  $k$  subgradient evaluations.
- 2 They also constructed convex quadratic functions in dimensions  $n \geq 2k$  where no first-order method can have function values more accurate than  $\mathcal{O}(\frac{1}{k^2})$  after  $k$  gradient evaluations.
- 3 For strongly convex functions with Lipschitz continuous gradients, the known lower bounds on the complexity allow a dimension independent linear rate of convergence  $\mathcal{O}(q^k)$  with  $0 < q < 1$ .




## 10. Perspective, open questions




- Convergence of the orbits for  $\alpha = 3$ ? of Nesterov algorithm?
- Convergence of the values: exhibit concrete examples showing that  $\alpha = 3$  is critical. Rate of convergence for  $a(t) = \frac{\alpha}{t}$ ,  $1 \leq \alpha < 3$ ?
- Find a Lyapunov function in the case  $\frac{1}{t^\gamma}$ , giving the rate of convergence  $\frac{1}{t^{1+\gamma}}$  for  $W(t)$ .
- Extend to the algorithmic part the convergence properties of the continuous dynamic (strong convergence...).
- Adaptive restart for (DIN-AVD), without strong convexity.
- Compare /combine with other rapid methods: multigrid, Newton based methods, other type of friction (dry).
- Show the  $\mathcal{O}(\frac{1}{t^2})$  convergence of the values for (DIN-AVD), combining Hessian driven and asymptotic vanishing damping.
- Extension to a non-convex setting: for analytic potentials, the convergence theory for HBF (HJ), and DIN (AABR) still works.
- Nonsmooth potentials, shock theory, PDE's hyperbolic equations.




*HAPPY BIRTHDAY*  
*TERRY*

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