# Limiting normal approach for quasiconvex analysis

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# Outline

**Alternative title:** A new step in quasiconvex calculus

- A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *quasiconvex* on $K$ if,

*for all $x, y \in K$ and all $t \in [0,1]$,*

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}.$$

or

*for all $\lambda \in \mathbb{R}$, the sublevel set*

$$S_\lambda = \{x \in X \ : \ f(x) \leq \lambda\} \textit{ is convex.}$$

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- $f$ differentiable
  $f$ is quasiconvex iff $df$ is quasimonotone
  $$\text{iff } df(x)(y-x) > 0 \Rightarrow df(y)(y-x) \geq 0$$

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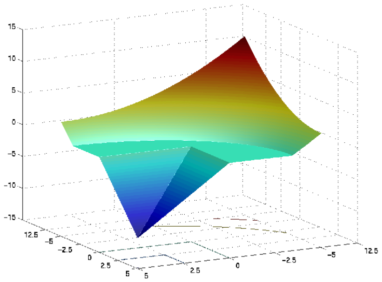
  $$\text{iff } df(x)(y-x) > 0 \Rightarrow df(y)(y-x) \geq 0$$
- $f$ is quasiconvex iff $\partial f$ is quasimonotone
  $$\text{iff } \exists\, x^* \in \partial f(x) \,:\, \langle x^*, y-x\rangle > 0$$
  $$\Rightarrow \forall\, y^* \in \partial f(y),\ \langle y^*, y-x\rangle \geq 0$$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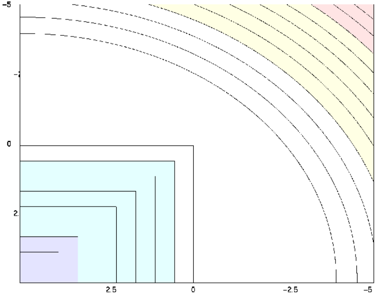
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- A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *semistrictly quasiconvex* on $K$ if,

*$f$ is quasiconvex and for any $x, y \in K$,*

$$f(x) < f(y) \Rightarrow f(z) < f(y), \quad \forall z \in [x, y[.$$

convex $\Rightarrow$ semistrictly quasiconvex $\Rightarrow$ quasiconvex

**Motivations :**

Our aim is to develop a “first order tool” for quasiconvex analysis/optimization that enjoy

- some generalized monotonicity
- some semicontinuity/closedness
- some sufficient optimality conditions (local or global)
- some calculus rules

# Outline

## Definition (Limiting sublevel set)

For a lower semicontinuous function $f : \mathbb{R}^m \to \mathbb{R}$ we define the *limiting sublevel set map* $S_f^l : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ as follows

$$S_f^l(\bar{x}) \equiv \operatorname*{Liminf}_{x \to \bar{x}} S_f(x), \quad \forall \bar{x} \in \mathbb{R}^m.$$

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Note the limiting sublevel set map $S_f^l$ is closed-valued by definition. We further observe that the lower limit in the above definition may be restricted as follows.

**Lemma**

*Let $f : \mathbb{R}^m \to \mathbb{R}$ be a lower semicontinuous function, then*

$$S_f^l(\bar{x}) = \operatorname*{Liminf}_{\substack{x \longrightarrow \bar{x} \\ S_f(\bar{x})}} S_f(x).$$

As for the classical sublevel set $S_f$ and strict sublevel set $S_f^<$, the convexity of the limit sublevel set characterizes the quasiconvexity of a lower semicontinuous function.

**Lemma (Characterization of quasiconvexity in terms of $S_f^l$)**

*Let $f : \mathbb{R}^m \to \mathbb{R}$ be a lower semicontinuous function. Then $f$ is quasiconvex if and only if $S_f^l(x)$ is convex for all $x \in \mathbb{R}^m$.*

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**Corollary (Inner semicontinuity of $S_f^l$)**

*For any lower semicontinuous function $f$, the limiting sublevel set map $S_f^l$ is inner semicontinuous.*

The limiting definition, inspired by the similar concept of limiting subdifferential, turns out to have the following very easy and natural equivalent explicit formulation for any lower semicontinuous function.

**Theorem (Explicit formula for $S_f^l(x)$)**

*Let $f$ be a lower semicontinuous function and $x \in \mathbb{R}^m$. Then*

$$S_f^l(x) = \begin{cases} \bar{S}_f^<(x) & \text{if } x \in \bar{S}_f^<(x), \\ S_f(x) & \text{otherwise.} \end{cases}$$

*In particular one always has $\bar{S}_f^<(x) \subset S_f^l(x) \subset S_f(x)$.*

### Definition (Limiting normal operator)

For quasiconvex lower semicontinuous function $f$ the *limiting normal operator* is a set-valued map $N^l_f : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ defined as

$$N^l_f(x) = (S^l_f(x) - x)^o, \quad \forall x \in \mathbb{R}^m.$$

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Note that for a quasiconvex function one has $\left(S_f^l(x) - x\right)^o = \left(T_{S_f^l(x)}(x)\right)^o$. Thus, the above introduced limiting normal operator is a local notion. An alternative definition of the limiting normal operator can be given in terms of upper limit of normal operator.

From classical subdifferential calculus (Clarke's book, e.g.):

- *if* $f$ *is Lipschitz on* $\mathbb{R}^m$
- *and* $0 \notin \partial f(x)$

*then*

$$cone\,(\partial f(x)) \subset N_f^l(x)$$

*and if* $f$ *is regular and semi-strictly quasiconvex,*

$$cone\,(\partial f(x)) = N_f(x).$$

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*then*

$$\mathit{cone}\,(\partial f(x)) \subset N^l_f(x)$$

*and if $f$ is regular and semi-strictly quasiconvex,*

$$\mathit{cone}\,(\partial f(x)) = N_f(x).$$

**Theorem (Relation of subdifferential and $N^l_f$)**

*For quasiconvex lower semicontinuous function* $f : \mathbb{R}^m \to \mathbb{R}$, *we have*

$$[\overline{\mathit{cone}}\,(\partial f(\bar{x})) \cup \partial^\infty f(\bar{x})] \subset N^l_f(\bar{x}),$$

*where equality holds provided* $0 \notin \partial f(\bar{x})$.

### Theorem

*For any quasiconvex lower semicontinuous function f it holds*

$$N_f^l(x) = \underset{x \to \bar{x}}{Limsup}\; N_f(x).$$

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*For any quasiconvex lower semicontinuous function $f$ it holds*

$$N_f^l(x) = \operatorname*{Limsup}_{x \to \bar{x}} N_f(x).$$

**Theorem**

*Let $f : \mathbb{R}^m \to \mathbb{R}$ be a lower semicontinuous quasiconvex function. Then*

1. $N_f^l$ *is quasimonotone,*
2. $N_f^l$ *is outer semicontinuous.*

Now consider the minimization problem

$$\min f(x) \qquad \text{subject to} \qquad x \in K, \tag{1}$$

where $K \subset \mathbb{R}^m$ is nonempty subset of $\mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}$ is quasiconvex. Then, necessary and sufficient optimality conditions may be stated as follows.

**Theorem (Necessary optimality conditions)**

*Let $f : \mathbb{R}^m \to \mathbb{R}$ be a lower semicontinuous quasiconvex function, $K$ be a nonempty convex set and $\bar{x} \in K$ be a solution to (1) such that $\bar{x} \in \bar{S}_f^<(\bar{x})$. Then it holds*

$$0 \in N_f^l(\bar{x}) \setminus \{0\} + N_K(\bar{x}).$$

**Theorem**

*Let $f : \mathbb{R}^m \to \mathbb{R}$ be a continuous quasiconvex function, $K$ be a nonempty subset of $\mathbb{R}^m$ and $\bar{x} \in K$. Then $\bar{x}$ a is local solution to (1) if one of the following hypothesis is satified:*

1. *the point $\bar{x}$ is a solution of the Stampacchia variational inequality defined by $N^l_f(\cdot) \setminus \{0\}$ and $K$;*
2. *the set $K$ is convex and $0 \in N^l_f(\bar{x}) \setminus \{0\} + N_K(\bar{x})$.*

**Theorem**

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Back to (local) paradise.....

**Corollary**

*Let $f : \mathbb{R}^m \to \mathbb{R}$ be a continuous quasiconvex function, $K$ be a nonempty convex set and $\bar{x} \in K$ be such that $\bar{x} \in \bar{S}_f^{<}(\bar{x})$. Then*

$$\bar{x} \in \arg\min_K f \quad \Leftrightarrow \quad 0 \in N_f^l(\bar{x}) \setminus \{0\} + N_K(\bar{x}).$$

**Corollary**

*Let $f : \mathbb{R}^m \to \mathbb{R}$ be a continuous quasiconvex function, $K$ be a nonempty convex set and $\bar{x} \in K$ be such that $\bar{x} \in \bar{S}_f^<(\bar{x})$. Then*

$$\bar{x} \in \arg\min_K f \quad \Leftrightarrow \quad 0 \in N_f^l(\bar{x}) \setminus \{0\} + N_K(\bar{x}).$$

Back to quasi-paradise.....

For quasiconvex functions there is two stable operations:

- Let $\mathcal{F} = \{f_i : \mathbb{R}^m \to \mathbb{R}\}_{i \in I}$ be a finite family of continuous quasiconvex functions and define

$$g(x) \equiv \max_{i \in I} f_i(x).$$

  Clearly $g$ is finite, quasiconvex and continuous on $\mathbb{R}^m$.

- For any function $g : \mathbb{R}^m \to \mathbb{R}$ defined as $g(x) = \theta(f(x))$ where $f$ is a lower semicontinuous quasiconvex function and $\theta : \mathbb{R} \to \mathbb{R}$ is an non-decreasing function

For any $x$, the index set $I(x)$ of active functions stands for $I(x) \equiv \{i \in I : f_i(x) = g(x)\}$.

**Theorem (Limiting normal operator to maximum)**

*Let $\mathcal{F} = \{f_i : \mathbb{R}^m \to \mathbb{R}\}_{1 \in I}$ be a finite family of continuous quasiconvex functions and $g$ be defined as above. Then*

$$N_g^l(\bar{x}) \subset \sum_{i \in I(\bar{x})} N_{f_i}^l(\bar{x})$$

*if the following constraint qualification is satisfied at $\bar{x} \in \mathbb{R}^m$*

$$\forall\, i \in I(\bar{x}) v_i \in N_{f_i}^l(\bar{x}) \quad \text{and} \quad \sum_{i \in I(\bar{x})} v_i = 0 \quad \Longrightarrow \quad \forall\, i \in I(\bar{x}) v_i = 0. \quad (2)$$

Let us recall that condition (2) means that the convex sets $\{S_{f_i}^l(\bar{x})\}_{i \in I(\bar{x})}$ can not be separated. Equivalently we may say that $\bar{x}$ is not an extremal point of the system $\{S_{f_i}^l(\bar{x})\}_{i \in I(\bar{x})}$, see Mordukhovich[Corollary 2.4 and Theorem 2.8]Mor06.

For any function $g : \mathbb{R}^m \to \mathbb{R}$ defined as $g(x) = \theta(f(x))$ where $f$ is a lower semicontinuous quasiconvex function and $\theta : \mathbb{R} \to \mathbb{R}$ is an increasing function then $N^l_g(x) = N^<_f(x)$ for any $x \in \mathbb{R}^m$ as could be easily verified from definitions. For the case of the composition with a (only) non-decreasing function the chain rule is as follows.

**Theorem (Chain rule for limiting normal operator)**

*Consider a lower semicontinuous quasiconvex function $f : \mathbb{R}^m \to \mathbb{R}$, a non-decreasing lower semicontinuous function $\theta : \mathbb{R} \to \mathbb{R}$ and their lower semicontinuous quasiconvex composition $g(x) = \theta(f(x))$. Then for any $\bar{x} \in \mathbb{R}^m$, the limiting normal operator $N^l_g(\bar{x})$ and strict normal operator $N^<_g(\bar{x})$ at point $\bar{x}$ satisfy*

$$\begin{array}{rcl} N^l_g(\bar{x}) & \subset & N^l_f(\bar{x}) \\ N^<_g(\bar{x}) & \supset & N^<_f(\bar{x}) \end{array}$$

*These inclusions become equalities, that is $N^l_g(\bar{x}) = N^l_f(\bar{x})$, provided $\bar{x} \in \bar{S}^<_g(\bar{x})$.*

# Outline

I- Introduction

II- Limiting normal operator

a- Definition

b- Main properties

III- And what about previous concepts?

Sublevel set:

$$S_\lambda = \{x \in X \ : \ f(x) \leq \lambda\}$$
$$S_\lambda^> = \{x \in X \ : \ f(x) < \lambda\}$$

Normal operator:

Define $N_f(x) : X \to 2^{X^*}$ by

$$\begin{aligned} N_f(x) &= N(S_{f(x)}, x) \\ &= \{x^* \in X^* \ : \ \langle x^*, y - x\rangle \leq 0, \ \forall y \in S_{f(x)}\}. \end{aligned}$$

With the corresponding definition for $N_f^>(x)$

*Seminal work of Borde-Crouzeix (1980)*

Adjusted sublevel set:

For any $x \in \operatorname{dom} f$, we define

$$S_f^a(x) = S_{f(x)} \cap \overline{B}(S_{f(x)}^<, \rho_x)$$

where $\rho_x = dist(x, S_{f(x)}^<)$, if $S_{f(x)}^< \neq \emptyset$.

Ajusted normal operator:

$$N_f^a(x) = \{x^* \in X^* \; : \; \langle x^*, y - x\rangle \leq 0, \;\; \forall y \in S_f^a(x)\}$$

*defined in D.A.-Hadjisavvas SIOPT (2005)*

# Final assessment

| Operator | Quasimono. | Closedness | Suff.Opt. cond. | Calculus rules |
|---|---|---|---|---|
| $N_f$ | Yes | No | No | |
| $N_f^<$ | No | Yes | No | |
| $N_f^a$ | Yes | Yes | Yes | |
| $N_f^l$ | Yes | Yes | Yes | Yes |

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- $N_f^>(x) = N(S_{f(x)}^>, x)$ has no quasimonotonicity properties

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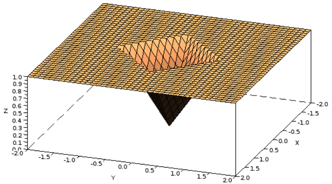
Example

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(a,b) = \begin{cases} |a| + |b|, & \text{if } |a| + |b| \leq 1 \\ 1, & \text{if } |a| + |b| > 1 \end{cases} .$$




Then $f$ is quasiconvex.

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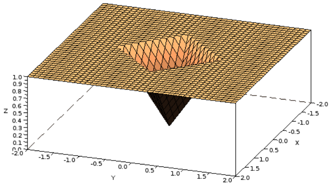


Then $f$ is quasiconvex.
Consider $x = (10,0)$, $x^* = (1,2)$, $y = (0,10)$ and $y^* = (2,1)$.

We see that $x^* \in N^{<}(x)$ and $y^* \in N^{<}(y)$ (since $|a|+|b| < 1$ implies $(1,2)\cdot(a-10,b) \le 0$ and $(2,1)\cdot(a,b-10) \le 0$)

while $\langle x^*, y-x \rangle > 0$ and $\langle y^*, y-x \rangle < 0$. Hence $N^{<}$ is not quasimonotone.

- $N_f(x) = N(S_{f(x)}, x)$ has no upper-semicontinuity properties
- $N_f^>(x) = N(S_{f(x)}^>, x)$ has no quasimonotonicity properties

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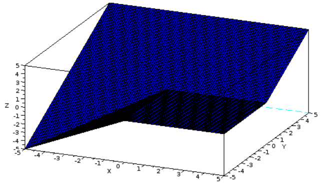


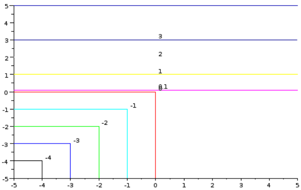


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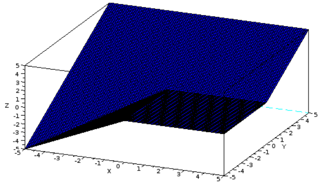


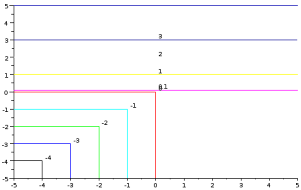

Then $f$ is quasiconvex.

We easily see that $N(x)$ is not upper semicontinuous....

# But ...another example

- $N_f(x) = N(S_{f(x)}, x)$ has no upper-semicontinuity properties
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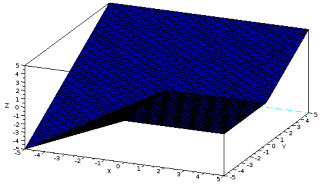


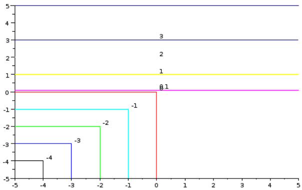


Then $f$ is quasiconvex.

We easily see that $N(x)$ is not upper semicontinuous....

These two operators are essentially adapted to the class of semi-strictly quasiconvex functions. Indeed in this case, for each $x \in \operatorname{dom} f \setminus \arg\min f$, the sets $S_{f(x)}$ and $S_{f(x)}^<$ have the same closure