

My first meeting with the mathematical work of Terry

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Brézis's question

"Understand the nonlinear σ -angle-boundedness with the cyclically monotonicity"

Definition

A is σ -angle-bounded if

$$(Ax - Ay, z - y) \leq \sigma(Ax - Ay, x - y)$$

A is n -cyclically monotone if

$$\sum_{i=1}^n (Ax_i, x_i - x_{i+1}) \geq 0 \text{ with } x_{n+1} = x_1$$

I met Terry for this occasion not the real person but THE ARTICLE :

Characterization of the subdifferentials of convex functions

Asplund's Theorem

Let A a (non-bounded) linear operator from $D(A) \subset E$ into E^* . We have the equivalence :

A is n -cyclically monotone

A is σ_n -angle-bounded with $\sigma_n = \frac{1}{4\cos^2(\pi/n)}$

Brézis :
"You should work with Georges Haddad..."

Proposition

Let E a normed vector space and E' the topological dual. Let $T : E \rightarrow E'$ a L -lipschitzian mapping which is σ -angle-bounded. We have the inequality :

$$(Tx - Ty, x - y) \geq \frac{1}{4\sigma L} \|Tx - Ty\|^2$$

Proof :

It is just a "good application" of the discriminant for a second degree equation

For non-linear operator :

σ -angle bounded
 \neq
n-cyclically monotone

Proposition

Let E a normed vector space and E' the topological dual.

Let $T : E \rightarrow E'$ a L -lipschitzian mapping which is n -cyclically monotone with smooth condition.

We have the inequality :

$$(Tx - Ty, x - y) \geq \frac{1}{4\sigma_n L} \|Tx - Ty\|^2 \text{ with } \sigma_n = \frac{1}{4\cos^2(\pi/n)}$$

Proof(Idea) :

$$Tx - Ty = \int_0^1 T'(v + t(x - y))(x - y) dt$$

T' is n -cyclically monotone and therefore σ -angle-bounded

THANKS TERRY

Terry's Theorem

When the space is a Banach space, the subdifferential ∂f of a l.s.c. proper convex function f is a maximal cyclically monotone operator and conversely.

Corollary

Under appropriate conditions, if ∂f is L-Lipschitzian

$$(\partial f(x) - \partial f(y), x - y) \geq \frac{1}{L} \|\partial f(x) - \partial f(y)\|^2$$

Convergence of the sequence $x^k = T x^{k-1}$ in an Hilbert space.

One hypothesis : *firmly non expansive*

$$(Tx - Ty, x - y) \geq \|Tx - Ty\|^2$$

Theorem(Byrne 2014)

Let $f : \mathcal{H} \rightarrow \mathbb{R}$ be convex and Gâteaux differentiable. The following are equivalent :

1. the function $F(x) = \frac{1}{2}\|x\|^2 - f(x)$ is convex.
2. the gradient operator $T = \nabla f$ is firmly non expansive
3. the function f is Fréchet differentiable and $T = \nabla f$ is non expansive

Appropriate conditions ? ? ? ?

Bauschke and Combettes (2010) shows for example :

Let C be a non empty open convex subset of an Hilbert Space \mathcal{H} , let $f : C \rightarrow \mathbb{R}$ be convex and twice continuously Fréchet differentiable on C , and let $\beta \in]0, \infty[$. Then ∇f is β -Lipschitz continuous if and only if it is $1/\beta$ -cocoercive.

Open Question : What is the "best" appropriate conditions ?

Happy Birthday Terry