On locally Lipschitz functions

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Overview

1. Classes of Lipschitz-like functions on metric spaces
2. Uniform approximation by Lipschitz-like functions
3. Other results

Outline

1. Classes of Lipschitz-like functions on metric spaces
2. Uniform approximation by Lipschitz-like functions
3. Other results

Definitions

Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be metric spaces, and let $f : X \to Y$.

- f is called λ-Lipschitz for $\lambda \in (0, \infty)$ if for all $\{x, w\} \subseteq X$, we have $\rho(f(x), f(w)) \leq \lambda d(x, w)$;
- f is called locally Lipschitz provided for each $x \in X$, there exists $\delta_x > 0$ such that the restriction of f to $S_d(x, \delta_x)$ is Lipschitz (where the local Lipschitz constant and δ_x depend on x);
- f is called uniformly locally Lipschitz if f is locally Lipschitz and δ_x can be chosen independent of x;
- f is called Lipschitz in the small if it is uniformly locally Lipschitz and the local Lipschitz constant can also be chosen independent of x, i.e., there exists $\delta > 0$ and $\lambda > 0$ such that $d(x, w) < \delta \Rightarrow \rho(f(x), f(w)) \leq \lambda d(x, w)$.

The notion of Lipschitz in the small function is due to J. Luukkainen [Luukkainen 1978-79] and later studied by G. Beer, M. I. Garrido, and J. Jaramillo in various combinations [Beer 1999, Beer-Garrido 2014, Beer-Garrido 2015, Garrido-Jaramillo 2008].

Examples

Example

$f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is locally Lipschitz but not uniformly locally Lipschitz.

Example

$f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is uniformly locally Lipschitz but not Lipschitz in the small.

Example

If $X = \cup_{n=1}^{\infty}[n - \frac{1}{4}, n + \frac{1}{4}]$ equipped with the usual metric of the line, then the locally constant function defined by $f(x) = n^2 \text{ if } n - \frac{1}{4} \leq x \leq n + \frac{1}{4}$ is Lipschitz in the small but fails to be Lipschitz on X.

Cauchy continuous function

Remark

Note that each locally Lipschitz function is continuous, each uniformly locally Lipschitz function maps Cauchy sequences in X to Cauchy sequences in Y, and each Lipschitz in the small function is uniformly continuous.

Functions that preserve Cauchy sequences are called Cauchy continuous and lie properly between the continuous functions and the uniformly continuous ones; we refer the reader to the expository article of Snipes [Snipes 1977] and the monograph of Lowen-Colebunders [Colebunders 1989].

Proposition

Let $\langle X, d \rangle$ and $\langle Y, \rho \rangle$ be metric spaces, and let $f : X \to Y$. Then f is Cauchy continuous if and only if f is uniformly continuous on each totally bounded subset of X.

- A real-valued f is is Lipschitz in the small if and only if $\exists \delta > 0, \lambda > 0$ such that whenever $(x, \alpha) \in \mathrm{epi}(f)$ and (x, β) satisfies $d(x, w) < \delta$ and $\beta \geq f(x) + \lambda d(x, w)$, then $(w, \beta) \in \mathrm{epi}(f)$.
- When X is a normed linear space, we are saying that $\mathrm{epi}(f)$ is stable under an epigraphical sum with λ times the norm plus the indicator function of δ times the unit ball. Such a "smoothing kernel" has an epigraph that looks like an inverted pencil whose point has slope λ and whose shaft has radius δ.

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A counter-example

Question: Can each continuous real-valued function be approximated by Lipschitz functions?

Answer: No, not even if the function is bounded!

Example

Let $X = \mathbb{N} \cup \{n + \frac{1}{n+1} : n \in \mathbb{N}\}$, equipped with the usual metric d of the line, and let f be the characteristic function of $\mathbb{N}$, that is, $f(x) = 1$ if $x \in \mathbb{N}$ and $f(x) = 0$ otherwise. Since the topology of X is discrete, f is continuous. Now if g satisfies $\sup_{x \in X} |f(x) - g(x)| < \frac{1}{4}$, then for each $n \in \mathbb{N}, g(n) > \frac{3}{4}$ while $g(n + \frac{1}{n+1}) < \frac{1}{4}$. This means that for each n,

$$\frac{|g(n) - g(n + \frac{1}{n+1})|}{d(n, n + \frac{1}{n+1})} > \frac{1/2}{\frac{1}{n+1}} = \frac{n+1}{2}.$$

The following classical result can be found in the 2001 monograph of J. Heinonen [Heinonen 2001].

Theorem

Each bounded uniformly continuous real-valued function on $\langle X, d\rangle$ can be uniformly approximated by Lipschitz functions.

An unbounded uniformly continuous real-valued may not be uniformly approximated by Lipschitz function: consider $x \mapsto x^2$ defined on $\mathbb{N}$.

That the locally Lipschitz functions are uniformly dense in the continuous real-valued functions on $\langle X, d\rangle$ follows most easily from the following result of Z. Frolik [1] regarding the existence of a "Lipschitz partition of unity" subordinated to any open cover of the metric space. He published his theorem in 1984, i.e., it is a surprisingly recent result.

Theorem

Let $\{V_j : j \in J\}$ *be an open cover of* $\langle X, d \rangle$*. Then there is a family of nonnegative real-valued Lipschitz functions* $\{p_i : i \in I\}$ *with the following properties:*

1. *for each* $x \in X$*, all but finitely many* p_i *vanish on some open ball with center* x*;*
2. *for each* $x \in X, \sum_{i \in I} p_i(x) = 1$*;*
3. $\forall i \in I, \exists j \in J$ *with* $\{x \in X : p_i(x) \neq 0\} \subseteq V_j$*.*

Theorem

Let $\langle X, d\rangle$ be a metric space. Then each continuous real-valued function f on X can be uniformly approximated by locally Lipschitz functions.

Proof.

- Let $\varepsilon > 0$, and for each $x \in X$ choose $\delta_x > 0$ such that $d(x, w) < \delta_x \Rightarrow |f(x) - f(w)| < \varepsilon$.
- Let $\{p_i : i \in I\}$ be a Lipschitz partition of unity subordinated to $\{S_d(x, \delta_x) : x \in X\}$, and for each $i \in I$, choose $x(i) \in X$ with $\{x : p_i(x) \neq 0\} \subseteq S_d(x(i), \delta_{x(i)})$.
- Then $\sum_{i \in I} f(x(i))p_i$ is a locally Lipschitz function that ε-approximates f in uniform distance.

□

A very different path to this result has been by taken by M. I. Garrido and J. Jaramillo in 2004 [Garrido-Jaramillo 2004]; there, it falls out of a general approximation theorem reminiscent of the program of M. H. Stone.

Theorem by Beer and Garrido

We now turn to the uniform approximation of arbitrary uniformly continuous real-valued functions. The following proof was recently presented by Beer and Garrido [Beer-Garrido 2015]; the result itself was first proved by Garrido and Jaramillo in a 2008 paper [Garrido-Jaramillo 2008].

Theorem

Let $\langle X, d \rangle$ be a metric space. Then each uniformly continuous real-valued function f on X can be uniformly approximated by Lipschitz in the small functions.

Proof of theorem by Beer and Garrido

Choose $\delta > 0$ such that for $x, w \in X$, $d(x, w) < \delta \Rightarrow |f(x) - f(w)| < \varepsilon$. Choose $k \in \mathbb{N}$ such that $\frac{k\delta}{2} > 1 + \varepsilon$. Define $g : X \to \mathbb{R}$ by

$$g(x) := \inf_{w \in S_d(x,\delta)} f(w) + kd(x, w).$$

Note that for each $x \in X$,

$$f(x) \geq g(x) \geq \inf_{w \in S_d(x,\delta)} f(w) \geq f(x) - \varepsilon.$$

One can show that if $d(x_1, x_2) < \frac{\delta}{2}$, then $|g(x_2) - g(x_1)| \leq kd(x_2, x_1)$.

Cofinally Cauchy sequence

In view of the two preceding results, we might expect to be able to uniformly approximate each Cauchy continuous real-valued function by uniformly locally Lipschitz functions. While a parallel result does not hold, Beer and Garrido [Beer-Garrido 2015] were able to characterize the domain spaces on which such uniform approximation of Cauchy continuous functions is possible.

Definition

A sequence (x_n) in $\langle X, d\rangle$ is called cofinally Cauchy provided for each $\varepsilon > 0$ there exists an infinite subset $\mathbb{N}_0$ of $\mathbb{N}$ such that whenever $\{n, j\} \subseteq \mathbb{N}_0$, we have $d(x_n, x_j) < \varepsilon$.

Theorem

Let $\langle X, d\rangle$ be a metric space. The following conditions are equivalent:

1. *each Cauchy continuous real-valued function f on X can be uniformly approximated by uniformly locally Lipschitz functions;*
2. *each cofinally Cauchy sequence in $\langle X, d\rangle$ has a Cauchy subsequence;*
3. *each cofinally Cauchy sequence in the completion of $\langle X, d\rangle$ has a convergent subsequence.*

Remark

It is not hard to produce a cofinally Cauchy sequence in any infinite dimensional normed linear space without a Cauchy subsequence, so that the approximation property in question fails in that setting.

Cofinally complete metric space

Metric spaces in which each cofinally Cauchy sequence has a convergent subsequence are called cofinally complete and form a well-studied class of spaces lying properly within the class of complete metric spaces. They have interesting characterizations in terms of covering properties (see, e.g., [Beer 2008, Hohti 1981, Rice 1977]).

Theorem

Let $\langle X, d\rangle$ *be a metric space. The following conditions are equivalent:*

1. $\langle X, d\rangle$ *is a cofinally complete space;*
2. *each continuous function* f *on* X *with values in a second metric space is uniformly locally bounded: there exists* $\mu > 0$ *such that* f *restricted to each ball of radius* μ *is bounded;*
3. *each continuous real-valued function on* X *is uniformly locally bounded;*
4. *for each open cover* $\mathscr{V}$ *of* X*, there exists* $\mu > 0$ *such each ball of radius* μ *in* X *has a finite subcover;*
5. *for each open cover* $\mathscr{V}$ *of* X*, there is an open refinement* $\mathscr{U}$ *and* $\mu > 0$ *such that for each* $x \in X, S_d(x, \mu)$ *intersects at most finitely many members of* $\mathscr{U}$*;*
6. $nlc(X) := \{x \in X : x \textit{ has no compact neighborhood}\}$ *is compact, and* $\forall \delta > 0\ \exists \lambda > 0$ *such that* $d(x, nlc(X)) > \delta \Rightarrow \{w \in X : d(x, w) \leq \lambda\}$ *is compact.*

By the second property, each uniformly locally compact space (such as $\mathbb{R}^n$) is cofinally complete. The fourth property above is weaker than requiring that each open cover of X has the property that each ball of a prescribed radius lies within a single member of the cover, that is, the cover has a Lebesgue number. The fifth property above is called uniform paracompactness. The cofinally complete spaces sit properly between the complete metric spaces and the compact metric spaces

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First result

One wishes to characterize those spaces $\langle X, d\rangle$ on which we have pairwise coincidence of our three classes of locally Lipschitz functions. The cofinally complete metric spaces arise in settling this general question (see [Beer-Garrido 2015]).

Theorem

Let $\langle X, d\rangle$ be a metric space. The following conditions are equivalent:

1. *$\langle X, d\rangle$ is a cofinally complete metric space;*
2. *each locally Lipschitz function on X with values in a metric space $\langle Y, d\rangle$ is uniformly locally Lipschitz;*
3. *each real-valued locally Lipschitz function on X is uniformly locally Lipschitz.*

Remark

- The last result is a little surprising, in that the class of continuous functions on $\langle X, d \rangle$ agrees with the class of Cauchy continuous functions if and only if $\langle X, d \rangle$ is a complete metric space;
- We have also characterized coincidence of the uniformly locally Lipschitz functions with the Lipschitz in the small functions, and coincidence of the locally Lipschitz functions with the Lipschitz in the small functions, but time does not permit a discussion.

Bornology

Given a family of functions $\mathscr{F}$ defined on a metric space $\langle X, d \rangle$ with values in one or more metric spaces, the family of subsets on which each member of $\mathscr{F}$ is bounded forms a bornology, that is, the family is stable under finite unions, contains the singletons, and is hereditary. Obviously, for the Lipschitz functions on $\langle X, d \rangle$ we get the bornology of d-bounded subsets. Given a metric space $\langle X, d \rangle$, we denote the bornology of its metrically bounded subsets by $\mathscr{B}_d(X)$.

For our three classes of locally Lipschitz functions, the common sets of boundedness are described by our final three results, as established by the authors in an earlier paper [Beer-Garrido 2014].

The common sets of boundedness: Theorem 1

Theorem

Let $\langle X, d\rangle$ be a metric space and let B be a nonempty subset. The following conditions are equivalent:

1. $cl(B)$ *is compact;*
2. *whenever* $\langle Y, \rho\rangle$ *is a metric space and* $f : X \to Y$ *is continuous, then* $f(B) \in \mathscr{B}_\rho(Y)$;
3. *whenever* $\langle Y, \rho\rangle$ *is a metric space and* $f : X \to Y$ *is locally Lipschitz, then* $f(B) \in \mathscr{B}_\rho(Y)$;
4. *whenever* $f : X \to \mathbb{R}$ *is locally Lipschitz, then* $f(B)$ *is a bounded set of real numbers.*

The common sets of boundedness: Theorem 2

Theorem

Let $\langle X, d\rangle$ be a metric space and let B be a nonempty subset. The following conditions are equivalent:

1. *B is d-totally bounded;*
2. *whenever $\langle Y, \rho\rangle$ is a metric space and $f : X \to Y$ is Cauchy continuous, then $f(B) \in \mathscr{B}_\rho(Y)$;*
3. *whenever $\langle Y, \rho\rangle$ is a metric space and $f : X \to Y$ is uniformly locally Lipschitz, then $f(B) \in \mathscr{B}_\rho(Y)$;*
4. *whenever $f : X \to \mathbb{R}$ is uniformly locally Lipschitz, then $f(B)$ is a bounded set of real numbers.*

Bourbaki bounded set

A subset A of $\langle X, d\rangle$ is called Bourbaki bounded if for each $\varepsilon > 0$ there is a finite subset F of X and $n \in \mathbb{N}$ such that each point of A can be joined to some point of F by an ε-chain of length at most n.
[Bourbaki 1966, Hejcman 1959, Vroegrijk 2009].

The common sets of boundedness: Theorem 3

Theorem

Let $\langle X, d\rangle$ be a metric space and let B be a nonempty subset. The following conditions are equivalent:

1. *B is d-Bourbaki bounded;*
2. *whenever $\langle Y, \rho\rangle$ is a metric space and $f : X \to Y$ is uniformly continuous, then $f(B) \in \mathscr{B}_\rho(Y)$;*
3. *whenever $\langle Y, \rho\rangle$ is a metric space and $f : X \to Y$ is Lipschitz in the small, then $f(B) \in \mathscr{B}_\rho(Y)$;*
4. *whenever $f : X \to \mathbb{R}$ is Lipschitz in the small, then $f(B)$ is a bounded set of real numbers.*

The last result was anticipated by a result of Atsuji [Atsuji 1958].

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