A note on the characterization of optimal allocations in OLG economies with multiple goods

J.-M. Bonnisseau and L. Rakotonindrainy[1]

[1] Paris School of Economics, Université Paris 1 Panthéon-Sorbonne
Centre d'Economie de la Sorbonne, UMR CNRS 8174

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Outline

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Introduction

A very standard question in optimization: first-order characterization of Pareto optimal allocation;

A non-standard framework: countably many objective functions;

A non-standard space: $\prod_{t=1}^{\infty} \mathbb{R}^{L_t}$ and the product topology.

A direct proof based on Balasko-Shell

Outline

A standard OLG exchange economy $\mathcal{E}$

Infinitely many dates ($t = 1, 2, \cdots$),

at date t,

- a finite set $\mathcal{L}_t$ of available commodities;

- a finite and non-empty set of consumers $\mathcal{I}_t$ living two periods, consumption set $X^i = \mathbb{R}_+^{L_t} \times \mathbb{R}_+^{L_{t+1}}$, endowments $e^i \in \mathbb{R}_{++}^{L_t} \times \mathbb{R}_{++}^{L_{t+1}}$;

- for each consumer i, a strict preference relation P_i from X^i to X^i.

Notations: $\bar{e}_t = \sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} e_t^i$

Assumption A.

a) For all i in $\mathcal{I}$, $P^i(x^i)$ *open in* X, *convex,*
$x^i \in \bar{P}^i(x^i)$ *and* $x^i \notin P^i(x^i)$,
$P^i(x^i) + (\mathbb{R}^{L_t}_+ \times \mathbb{R}^{L_{t+1}}_+) \subset P^i(x^i)$.

b) For all $i \in \mathcal{I}$, *for all* x_i *in the interior of* X_i,
$-N_{\bar{P}^i(x^i)}(x^i)$ *is a half line* $\{t\gamma^i(x^i) \mid t \geq 0\}$,
$\gamma^i(x^i) \in \mathbb{R}^{L_t}_{++} \times \mathbb{R}^{L_{t+1}}_{++}$ *continuous mapping on the interior of* X^i, $\|\gamma^i(x^i)\| = 1$.

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Pareto optimal allocations

Definition

x feasible is Pareto optimal (PO) (resp. weakly Pareto optimal (WPO)) if there is no (y^i) in $\prod_{i \in \mathcal{I}} X^i$ such that:

$$\sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} y^i_t = \bar{e}_t, \text{ for } t \geq 1$$

and for all $i \in \mathcal{I}$, $y^i \in \bar{P}^i(x^i)$, with $y^i \in P^i(x^i)$ for at least one individual i (resp. $\exists \bar{t}$ such that $\forall t \geq \bar{t}$, $\forall i \in \mathcal{I}_t$ $y^i = x^i$).

Characterization of WEAK Pareto optimal allocations

Definition

A feasible allocation $x = (x^i)$ is supported by the price p if for each $t \in \mathbb{N}$, for all $i \in \mathcal{I}_t$ and for all $\xi^i \in P^i(x^i)$, $\Pi_t \cdot \xi^i > \Pi_t \cdot x^i$, where $\Pi_t = (p_t, p_{t+1})$.

Lemma

The feasible interior allocation $x = (x^i)$ is WPO if and only if there exists a price sequence p which supports $x = (x^i)$. Then Π_t is collinear to $\gamma^i(x^i)$ for all t and $i \in \mathcal{I}_t$.

Additional assumptions

$x = (x^i) \in \prod_{i \in \mathcal{I}} X^i$ *a feasible allocation:* **Assumption C:** $\exists \underline{r} > 0, \forall i \in \mathcal{I}, \bar{B}_{t,t+1}(x^i + \underline{r}\gamma^i(x^i), \underline{r}) \setminus \{x^i\} \subset P^i(x^i)$;

Assumption C': $\exists \bar{r} > 0$, *for all* $i \in \mathcal{I}, \forall \xi^i \leq (\bar{e}_t, \bar{e}_{t+1})$ *if* $\xi^i \in P^i(x^i)$, *then* $\xi^i \in \bar{B}(x^i + \bar{r}\gamma^i(x^i), \bar{r})$;

Assumption G: $\exists \overline{\eta} > \underline{\eta} > 0, \forall i \in \mathcal{I}_{-0}, \underline{\eta} \leq \frac{\|\gamma_t^i(x^i)\|}{\|\gamma_{t+1}^i(x^i)\|} \leq \overline{\eta}$;

Assumption B: $\exists \bar{e} \in \mathbb{R}^L_{++}, \exists \underline{\varepsilon} > 0, \forall t \in \mathbb{N}^*, \bar{e}_t \leq \bar{e}$ *and* $\forall i \in \mathcal{I}_{-0}, (\underline{\varepsilon}\mathbf{1}_t, \underline{\varepsilon}\mathbf{1}_{t+1}) \leq x^i$.

Proposition

$x = (x^i) \in \prod_{i \in \mathcal{I}} X^i$ an allocation supported by $p = (p_1, p_2, \ldots p_t, \ldots)$ satisfying Assumptions A, C, C', G and B. Then, x is Pareto optimal if and only if:

$$\sum_{t \in \mathbb{N}^*} \frac{1}{\|p_t\|} = +\infty.$$

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Aggregation: one consumer at each generation

$$\bar{P}_t((x^i)) := \sum_{i \in \mathcal{I}_t} \bar{P}^i(x^i)$$

$\bar{x}^t := \sum_{i \in \mathcal{I}_t} x^i$

$$N_{\bar{P}_t((x^i))}(\bar{x}^t) = \bigcap_{i \in \mathcal{I}_t} N_{\bar{P}^i(x^i)}(x^i)$$

x Pareto optimal if and only if $\bar{x}$ Pareto optimal for the preference relation $\bar{P}_t$. x supported by a price, then $\bar{x}$ supported by the same price.

A geometric approach

Feasible agregate transfer: $\bar{h} = (\bar{h}^t_t, \bar{h}^t_{t+1})$ with $\bar{h}^t_{t+1} = -\bar{h}^{t+1}_{t+1}$.

Net present value: $\eta^t = \Pi_t \cdot \bar{h}^t$

$\alpha^t = \frac{\|\bar{h}^t\|^2 \|\Pi_t\|}{\eta^t}$

Lemma

Under Assumption C′, $\bar{h}$ Pareto improving implies α is bounded above.

Lemma

$\bar{h}$ feasible Pareto improving aggregate transfer, then:

$$\begin{aligned}\alpha^t &\geq \frac{\|\Pi_t\|}{\eta^t}\|\bar{h}^t\|^2 \\ &= \frac{\|\Pi_t\|}{\eta^t}\left[\frac{1}{\|p_t\|^2}(\eta^0+\ldots+\eta^{t-1})^2+\frac{1}{\|p_{t+1}\|^2}(\eta^0+\ldots+\eta^t)^2\right]\end{aligned}$$

Lemma

η a positive sequence in $\mathbb{R}$.

$$\bar{h}^t_{t+1} = \left(\eta^0 + \eta^1 + \ldots + \eta^t\right) \frac{p_{t+1}}{\|p_{t+1}\|^2} \text{ and } \bar{h}^t_t = -\bar{h}^{t-1}_t$$

$$\begin{aligned} \alpha^t &= \frac{\|\Pi_t\|}{\eta^t} \|\bar{h}^t\|^2 \\ &= \frac{\|\Pi_t\|}{\eta^t} \left[\frac{1}{\|p_t\|^2}(\eta^0 + \ldots + \eta^{t-1})^2 + \frac{1}{\|p_{t+1}\|^2}(\eta^0 + \ldots + \eta^t)^2\right] \end{aligned}$$

Under Assumptions B and C, if α bounded then $\exists \mu > 0$ such $\mu \bar{h}$ is a feasible Pareto improving aggregate transfer.