Terry Fest, May 2015

Level Set Methods in Convex Optimization

Objective-Constraint Interchange Methods

"Flippy"

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Thank you Samir, Vincent, and Henri!

Problem Framework

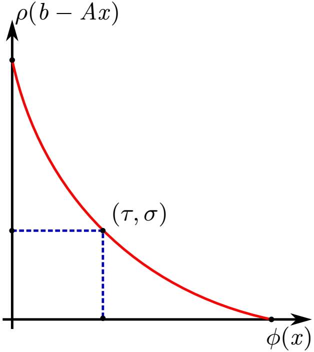
Assume ρ and ϕ are closed, proper, and convex

$$\begin{array}{|llll|}\hline \mathcal{P}_1(\sigma): & \min & \phi(x) & \text{st} \quad \rho(b-Ax) \le \boldsymbol{\sigma} \\ \mathcal{P}_2(\tau): & \min & \rho(b-Ax) & \text{st} \quad \phi(x) \le \boldsymbol{\tau} \\ \hline \end{array}$$

$\mathcal{P}_1(\sigma)$ is the target problem
$\mathcal{P}_2(\tau)$ is the easier flipped problem.

Problems $\mathcal{P}_1(\sigma)$ and $\mathcal{P}_2(\tau)$ are linked by

$$v_2(\tau) := \min_x \rho(b-Ax) + \delta\left((x,\tau) \,|\, \mathrm{epi}(\phi)\right)$$



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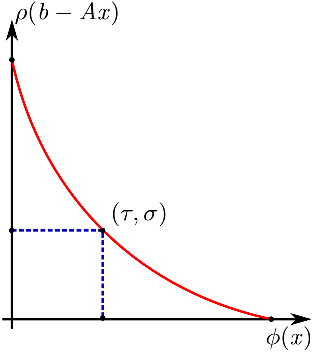

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Broad summary of results:

1. $v_2(\tau)$ is always convex (inf-projection), but may not be differentiable.
2. Solve $v_2(\tau) = \sigma$ by an inexact secant (Newton) method.
3. We have precise knowledge of the variational properties of $v_2(\tau)$ for a large classes of problems $\mathcal{P}_2(\tau)$.

Basic Definitions

The **horizon function** of h

$$h^\infty(z) := \sup_{x \in \mathrm{dom}(h)} \left[h(x+z) - h(x) \right].$$

The **perspective function** of h

$$\tilde{h}(z,\lambda) := \begin{cases} \lambda h(\lambda^{-1} z) & \text{if} \quad \lambda > 0, \\ \delta\left(x \mid 0\right) & \text{if} \quad \lambda = 0, \\ +\infty & \text{if} \quad \lambda < 0. \end{cases}$$

The **closure of the perspective function** of h

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Support Functions for $\mathrm{epi}(h)$ and $\mathrm{lev}_h(\tau)$ (Rockafellar 1966)

$h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be closed proper and convex.

Then

$$\delta^*\left((y,\mu) \mid \mathrm{epi}(h)\right) = (h^*)^\pi(y,-\mu)$$

and

$$\delta^*\left(y \mid \mathrm{lev}_h(\tau)\right) = \mathrm{cl}\left(\inf_{\mu \geq 0}\left[\tau\mu + (h^*)^\pi(y,\mu)\right]\right),$$

where

$$\begin{aligned}
\mathrm{epi}(h) &:= \{(x,\mu) \mid h(x) \leq \mu\} \\
\mathrm{lev}_h(\tau) &:= \{x \mid h(x) \leq \tau\} \\
\delta^*\left(z \mid C\right) &:= \sup_{w \in C} \langle z,\ w\rangle
\end{aligned}$$

Perturbation Framework and Duality (Rockafellar (1970))

The perturbation function

$$f(x, b, \tau) := \rho(b - Ax) + \delta\left((x, \tau) \mid \mathrm{epi}(\phi)\right)$$

Its conjugate

$$f^*(y, u, \mu) = (\phi^*)^\pi(y + A^T u, -\mu) + \rho^*(u)\ .$$

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The Primal Problem

$$\mathcal{P}(b, \tau): \qquad v(b, \tau) := \min_x f(x, b, \tau) \; .$$

The Dual Problem

$$\mathcal{D}(b, \tau): \qquad \hat{v}(b, \tau) \;:=\; \sup_{u,\mu} \langle b,\ u\rangle + \tau\mu - f^*(0, u, \mu)$$

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The Subdifferential: If $(b, \tau) \in \mathrm{int}\,(\mathrm{dom}(v))$, then $v(b, \tau) = \hat{v}(b, \tau)$ and

$$\emptyset \neq \partial v(b, \tau) = \operatorname*{arg\,max}_{u,\mu}\ \mathcal{D}(b, \tau)$$

The Constraint Qualification $(b,\tau) \in \operatorname{int}(\operatorname{dom}(v))$

The Primal Problem

$$\mathcal{P}(b,\tau): \qquad v(b,\tau) := \min_x \rho(b - Ax) + \delta\left((x,\tau) \mid \operatorname{epi}(\phi)\right)$$

The Constraint Qualification (Slater CQ)

$$0 \in \operatorname{int}(\operatorname{dom}(v)) \iff \exists\ \hat{x} \text{ st } \phi(\hat{x}) < \tau \text{ and } b - A\hat{x} \in \operatorname{int}(\operatorname{dom}(\rho))$$

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Solution Existence: Coercivity Conditions

- The dual objective is coercive iff $b \in \operatorname{int}(\operatorname{dom}(\rho) + A(\operatorname{lev}_\phi(\tau)))$.

- The primal objective is coercive iff

$$\operatorname{hzn}(\phi) \cap [-A^{-1}\operatorname{hzn}(\rho)] = \{0\}\ ,$$

where

$$\operatorname{hzn}(p) := \{y \mid p^\infty(y) \le 0\} = [\operatorname{lev}_p(\tau)]^\infty \quad \forall\ \tau > \inf p$$

KKT Conditions and Subdifferential Representations

KKT Conditions:

$$\bar{u} \in \partial\rho(b - A\bar{x}) \quad \text{and} \quad A^T\bar{u} \in N\left(\bar{x} \,|\, \mathrm{lev}_\phi(\tau)\right)$$

Subdifferential Representation 1: If $\partial v(b, \tau) \neq \emptyset$, then

$$\partial v(b,\tau) = \left\{ \begin{pmatrix} \bar{u} \\ -\bar{\mu} \end{pmatrix} \;\middle|\; \begin{array}{c} (\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m \text{ satisfy KKT cond. and} \\ \bar{\mu} \in \arg\min_{\mu \geq 0}[\tau\mu + (\phi^*)^\pi(A^T\bar{u}, \mu)] \end{array} \right\}.$$

Subdifferential Representation 2: If $\partial v(b, \tau) \neq \emptyset$ and $\mathrm{cone}\,(\partial\phi(\bar{x}))$ is closed for all $\bar{x} \in \arg\min_x f(x, b, \tau)$, then

$$\partial v(b,\tau) = \left\{ \begin{pmatrix} \bar{u} \\ -\bar{\mu} \end{pmatrix} \;\middle|\; \begin{array}{c} \exists\, \bar{x} \text{ s.t. } 0 \in -A^T\partial\rho(b - A\bar{x}) + \bar{\mu}^+\partial\phi(\bar{x}) \\ \text{with } \bar{\mu} \geq 0 \text{ and } \bar{\mu}(\phi(\bar{x}) - \tau) = 0 \end{array} \right\}.$$

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$$\partial v_2(\tau) = \left\{ \omega \;\middle|\; \exists\, u \text{ st } \begin{pmatrix} u \\ \omega \end{pmatrix} \in \partial v(b,\tau) \right\}$$

Piecewise Linear-Quadratic Penalties (Rockafellar 1988)

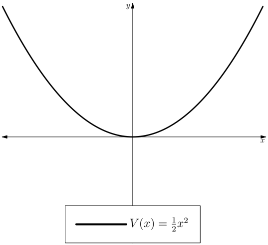
$$\phi(x) := \sup_{u \in U} \left[\langle x,\, u \rangle - \tfrac{1}{2} u^T B u \right]$$

$U \subset \mathbb{R}^n$ is nonempty, closed and convex with $0 \in U$ (not nec. poly.)
$B \in \mathbb{R}^{n \times n}$ is symmetric positive semi-definite.

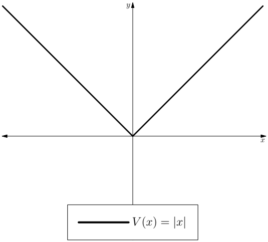
Examples:

1. Support functionals: $B = 0$
2. Gauge functionals: $\gamma\left(\cdot \mid U^\circ\right) = \delta^*\left(\cdot \mid U\right)$
3. Norms: $\mathbb{B} =$ closed unit ball, $\|\cdot\| = \gamma\left(\cdot \mid \mathbb{B}\right)$
4. Least-squares: $U = \mathbb{R}^n$, $B = I$
5. Huber: $U = [-\epsilon, \epsilon]^n$, $B = I$

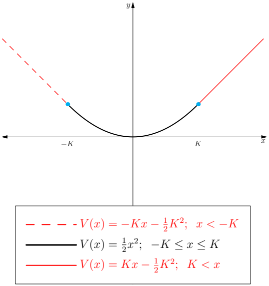
PLQ Densities: Gauss, Laplace, Huber, Vapnik



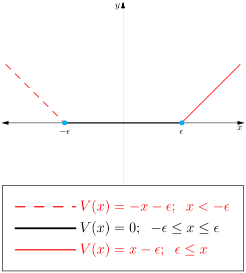
Gauss



Laplace



Huber



Vapnik

$$\phi(x) := \sup_{u \in U} \left[\langle x,\ u \rangle - \tfrac{1}{2} u^T B u \right]$$

$$\mathcal{P}(b,\tau): \qquad v(b,\tau) := \min \rho(b - Ax) \quad \text{st } \phi(x) \le \tau$$

$$\partial v(b,\tau) = \left\{ \begin{pmatrix} \bar{u} \\ -\bar{\mu} \end{pmatrix} \;\middle|\; \begin{array}{c} (\bar{x},\bar{u}) \text{ satisfy the KKT cond. for } \mathcal{P}(b,\tau) \text{ and} \\ \bar{\mu} = \max\left\{ \gamma\left(A^T\bar{u} \mid U\right), \sqrt{\bar{u}^T A B A^T \bar{u}}/\sqrt{2\tau} \right\} \end{array} \right\}.$$

A Few Special Cases

$$\boxed{\mathsf{v}_2(\tau) := \min \tfrac{1}{2}\|b - Ax\|_2^2 \quad \text{st } \phi(x) \le \tau}$$

Optimal Solution: $\bar{x}$
Optimal Residual: $\bar{r} = A\bar{x} - b$

1. **Support functionals:** $\phi(x) = \delta^*\left(x \mid U\right),\ 0 \in U \Longrightarrow$
   $v_2'(\tau) = -\delta^*\left(A^T\bar{r} \mid U^\circ\right) = -\gamma\left(A^T\bar{r} \mid U\right)$

2. **Gauge functionals:** $\phi(x) = \gamma\left(x \mid U\right),\ 0 \in U \Longrightarrow$
   $v_2'(\tau) = -\gamma\left(A^T\bar{r} \mid U^\circ\right) = -\delta^*\left(A^T\bar{r} \mid U\right)$

3. **Norms:** $\phi(x) = \|x\| \implies v_2'(\tau) = -\|A^T\bar{r}\|_*$

4. **Huber:** $\phi(x) = \sup\limits_{u \in [-\epsilon,\epsilon]^n} [\langle x,\ u\rangle - \tfrac{1}{2}u^Tu] \implies$
   $v_2'(\tau) = -\max\{\epsilon\|A^T\bar{r}\|_\infty,\ \|A^T\bar{r}\|_2/\sqrt{2\tau}\}$

5. **Vapnik:** $\phi(x) = \|(x-\epsilon)_+\|_1 + \|(-x-\epsilon)_+\|_1 \implies$
   $v_2'(\tau) = -(\|A^T\bar{r}\|_\infty + \epsilon\|A^T\bar{r}\|_1)$

Solve $v_2(\tau) = \sigma$ via Inexact Secant

Definition [Inexact function evaluation oracle]
Given $f : \mathbb{R} \to \mathbb{R}$*, an inexact evaluation oracle is a mapping* $\mathcal{O}_v$ *assigning to each pair* $(x, \alpha) \in \mathbb{R} \times (1, \infty)$ *real numbers* $(l, u) \in \mathbb{R} \times \mathbb{R}$ *satisfying*

$$l \leq f(x) \leq u \quad \textit{and} \quad 1 \leq \frac{u}{l} \leq \alpha.$$

Secant Input Data:
A non increasing convex function $v : \mathbb{R} \to \mathbb{R}$ *and an associated inexact evaluation oracle* $\mathcal{O}_v$*, a target accuracy* $\beta > 0$*, initial points* $t_0, t_1 \in \mathbb{R}$ *with* $t_0 < t_1$ *and* $v(t_0) \geq v(t_1) > 0$*, and a constant* $\alpha \in (1, 2)$*.*

An Inexact Secant Method

Initialization: Set $(l_0, u_0) = \mathcal{O}_v(t_0, \alpha)$, $(l_1, u_1) = \mathcal{O}_v(t_1, \alpha)$ and $k = 1$.

Iteration:

While $u_k > \beta$

if $u_k > u_{k-1}$

$u_k \leftarrow u_{k-1}$

if $l_k > l_{k-1}$

$l_k \leftarrow l_{k-1}$

end-if:

Set $\widehat{m}_k := \frac{u_{k-1} - l_k}{t_{k-1} - t_k}$ and $t_{k+1} := t_k + \frac{l_k}{-\widehat{m}_k}$.

Compute lower and upper bounds $(l_{k+1}, u_{k+1}) := \mathcal{O}_v(t_{k+1}, \alpha)$.

Set $k \leftarrow k + 1$.

end-while.

Super-Linear Convergence

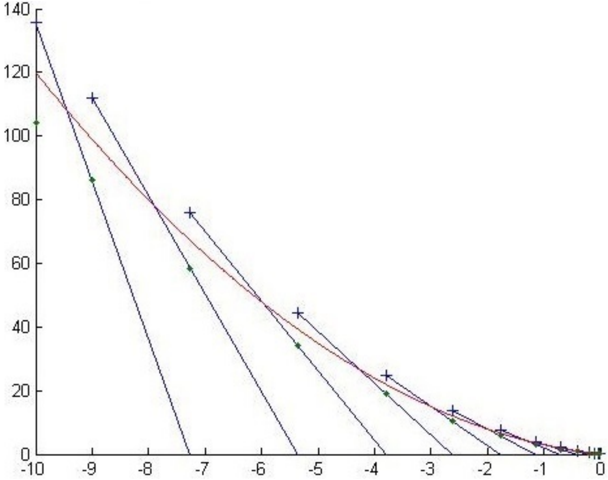
Suppose there exists $\bar{t} > t_1$ *such that*

$$v(\bar{t}) = 0 \quad \textit{and} \quad \partial v(\bar{t}) \cap (-\infty, 0) \neq \emptyset.$$

If the inexact secant method is implemented with $\beta = 0$*, then*

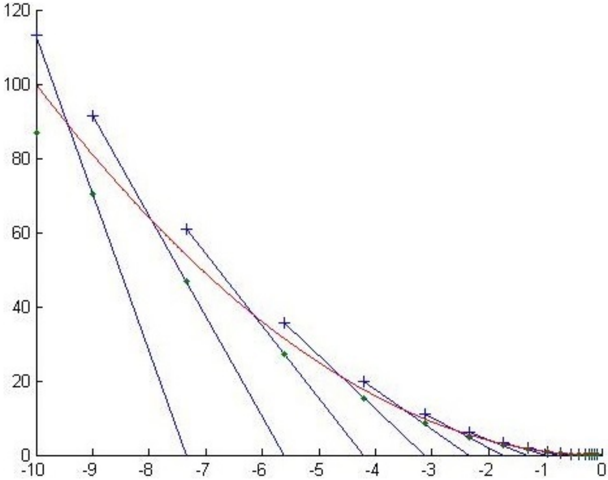
$$t_k \to \bar{t} \quad \textit{at a superlinear rate.}$$

Illustration



$f(t) = (t-1)^2 - 1$ with $\alpha = 1.3$, $\beta = 0.01$, $\#\text{iterations} = 16$.

Illustration: Degenerate Case



$f(t) = t^2$ with $\alpha = 1.3$, $\beta = 0.01$, $\#\text{iterations} = 18$.

Linear convergence of the inexact secant method

Theorem: The inexact secant method terminates (i.e., $u_K \le \beta$) after at most

$$K \le \max\left\{\log_{2/\alpha}\left(\frac{|\hat{m}_1|R}{\beta}\right) + \log_{2/\alpha}(2)\cdot\log_{2/\alpha}\left(\frac{2l_1}{\beta}\right) + 1,\ 2\right\}$$

iterations, where we set $R = \bar{t} - t_1$ (even in the degenerate case).

In particular, in the case $\alpha = \sqrt{2} \approx 1.41$, the iteration bound simplifies:

$$K \le \max\left\{\log_{\sqrt{2}}\left(\frac{l_1^2|\hat{m}_1|R}{\beta^3}\right) + 5,\ 2\right\}.$$

The Primal Problem

$$\mathcal{P}(b,\tau): \qquad \min_x \rho(b - Ax) + \delta\left((x,\tau) \mid \mathrm{epi}(\phi)\right) \ .$$

The Dual Problem

$$\mathcal{D}(b,\tau): \qquad \sup_u \langle b,\ u\rangle - \rho^*(u) - \delta^*\left(A^T u \mid \mathrm{lev}_\phi(\tau)\right) \ =: h(u,\tau).$$

$h(u,\tau)$ is concave in u and convex in τ.

Let $(\bar{u},\bar{\tau}) \in \mathrm{dom}(h)$ and let $\bar{s} \in \partial_\tau h(\bar{u},\bar{\tau})$, then, by weak duality, the line

$$\tau \mapsto h(\bar{u},\bar{\tau}) + \bar{s}(\tau - \bar{\tau})$$

is a lower minorant of the value function $v(\tau)$.

Inexact Newton Method

Consider any two points $\hat{\tau} < t\bar{\tau}$ along with lower and upper bounds,

$$\hat{l} \le v(\hat{\tau}) \le \hat{u} \qquad \text{and} \qquad \bar{l} \le v(\bar{\tau}) \le \bar{u}.$$

Suppose we are given $(\bar{u}, \bar{\tau}) \in \mathrm{dom}(h)$ satisfying $\bar{l} = h(\bar{u}, \bar{\tau})$.
Then the root of the secant line

$$l_s(\tau) := \bar{l} + \frac{\hat{u} - \bar{l}}{\hat{\tau} - \bar{\tau}}(\tau - \bar{\tau})$$

is no greater than the root of the Newton line

$$l_N(\tau) = \bar{l} + \bar{s}(\tau - \bar{\tau})$$

for any $\bar{s} \in \partial_\tau h(\bar{u}, \bar{\tau})$ (use subdifferential formulas for PLQ functions).

Sparse and Robust Formulation

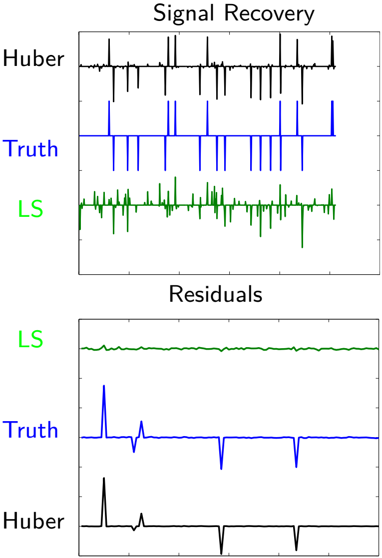
$$\boxed{\text{HBP}_\sigma: \quad \min \quad \|x\|_1 \quad \text{st} \quad \rho(b - Ax) \leq \boldsymbol{\sigma}}$$

Problem Specification

x	20-sparse spike train in $\mathbb{R}^{512}$
b	measurements in $\mathbb{R}^{120}$
A	Measurement matrix satisfying RIP
ρ	Huber function
σ	error level set at $.01$
5	outliers

Results

In the presence of outliers, the robust formulation recovers the spike train, while the standard formulation does not.



Sparse and Robust Formulation

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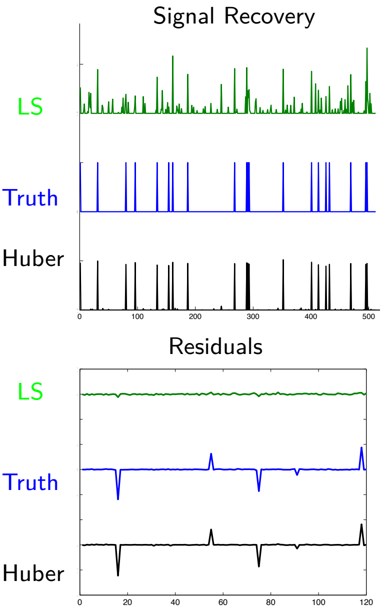

Least-Squares, Huber, and Student's t

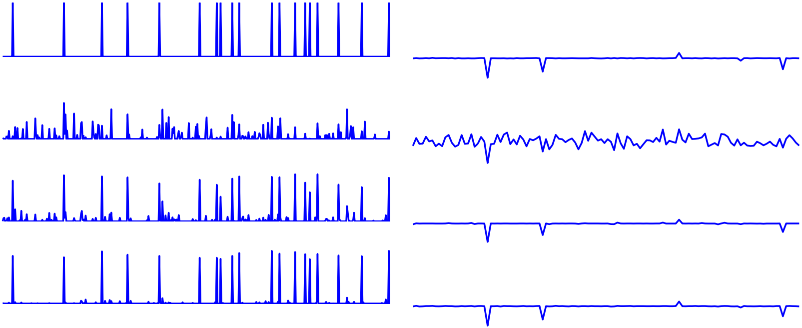

Figure : Left, top to bottom: True signal, and reconstructions via least-squares, Huber, and Student's t. Right, top to bottom: true errors, and least-squares, Huber, and Student's t residuals.

Homework

Go to the course homework webpage for the Huber-Laplace-Student t comparison for non RIP data.

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Thank you Terry for being my adventure guide both in mathematics and the wilderness!