Fixed-Point Iterations for Nonexpansive Maps
The sharp bound of asymptotic regularity is $1/\sqrt{\pi}$

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Based on joint work with

J-B. Baillon, M. Bravo, J. Soto, J. Vaisman

Celebrating Terry Rockafellar's 80th birthday

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Banach-Picard fixed point iteration

$T : X \to X$ contraction

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$$\|x^{n+1} - x^n\| = \|Tx^n - x^n\| \leq \rho^n \|Tx^0 - x^0\| \to 0$$

$$\Downarrow$$

convergence + error estimates + stopping rule

Krasnoselskii-Mann fixed point iteration

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(KM) $$\boxed{x^{n+1} = (1-\alpha_{n+1})x^n + \alpha_{n+1} T x^n}$$

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- algorithm for computing fixed points
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Question: $\|Tx^n - x^n\| \to 0$? (Asymptotic Regularity)

History of $\|Tx^n - x^n\| \to 0$

- 1955 **Krasnoselskii:** X uniformly convex, C compact, $\alpha_n \equiv 1/2$
- 1957 **Schaefer:** extends Krasnoselskii to $\alpha_n \equiv \alpha$
- 1966 **Ishikawa**: extends Krasnoselskii to X strictly convex
- 1966 **Browder-Petryshyn**: X uniformly convex, $\mathrm{Fix}(T) \neq \phi$, $\alpha_n \equiv \alpha$
- 1976 **Ishikawa**: X general, C compact, $\sum \alpha_n = \infty$, $\alpha_n \leq 1 - \epsilon$
- 1978 **Edelstein-O'Brien**: $\|Tx^n - x^n\| \to 0$ uniformly w.r.t x_0
- 1983 **Goebel-Kirk**: uniformly w.r.t. x_0 and T (for C given)
- 1992 **Baillon-Bruck**: $\alpha_n \equiv \alpha \Rightarrow \|Tx^n - x^n\| = O(1/\log n)$
- 2003 **Kohlenbach**: $\|Tx^n - x^n\| \to 0$ only depends on $diam(C)$

Baillon-Bruck's conjecture (1992)

There exists a universal constant κ such that

$$(BB) \qquad \|Tx^n - x^n\| \le \kappa \frac{\text{diam}(C)}{\sqrt{\sum_{k=1}^n \alpha_k(1-\alpha_k)}}$$

REMARK: in continuous time $\|Tx(t) - x(t)\| \le \kappa \frac{\text{diam}(C)}{\sqrt{t}}$

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Our contributions:

- **The bound holds for general α_n with $\kappa = 1/\sqrt{\pi} \sim 0.5642$**
- **Nonlinear maps: the constant $\kappa = 1/\sqrt{\pi}$ is optimal**
- **Affine maps: sharp bound with $\kappa = \max_z \sqrt{z}e^{-z}I_0(z) \sim 0.4688$**

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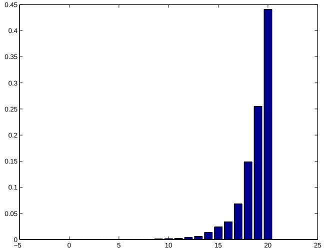
Recall

$$x^{n+1} = (1 - \alpha_{n+1})x^n + \alpha_{n+1} Tx^n$$

Alternative expression for x^n

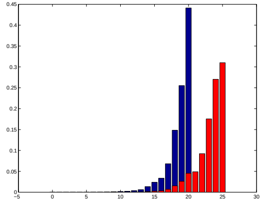
Let $\pi_i^n = \alpha_i \prod_{i+1}^n (1-\alpha_k)$ and set $Tx^{-1} = x_0$ by convention, then

$$x^n = \sum_{i=0}^n \pi_i^n Tx^{i-1}$$



A recursive bound $\|x^m - x^n\| \leq d_{mn}$

$$x^m - x^n = \sum_{j=0}^{m} \pi_j^m Tx^{j-1} - \sum_{i=0}^{n} \pi_i^n Tx^{i-1}$$



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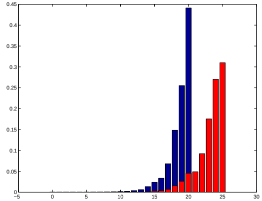

Let P_{mn} be the set of transport plans $z \ge 0$ taking π^n to π^m

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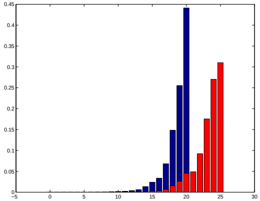

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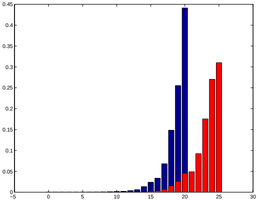

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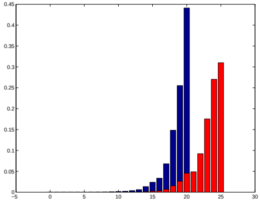

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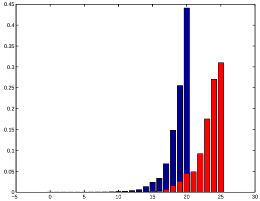
$$\|x^m - x^n\| \leq \sum_{j=0}^{m} \sum_{i=0}^{n} z_{ji} d_{j-1,i-1} \quad \longleftarrow \quad \min_{z}$$



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Set $d_{-1,n} = 1, d_{-1,-1} = 0$, and define inductively

$$(R) \qquad d_{mn} \triangleq \min_{z \in P_{mn}} \sum_{j=0}^{m} \sum_{i=0}^{n} z_{ji} d_{j-1,i-1}$$

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Theorem (Bravo-C. – Sep. 2014)

Bound (R) is best possible: There exists a non-expansive T on the set $C = [0,1]^{\mathbb{N}} \subseteq \ell^\infty(\mathbb{N})$ which attains $\|x^m - x^n\| = d_{mn}$ for all m, n.

Proof: Built from dual solutions of the optimal transports.

Restatement of (BB)

$$\|Tx^n - x^n\| = \|\tfrac{x^{n+1}-x^n}{\alpha_{n+1}}\| \leq \frac{d_{n,n+1}}{\alpha_{n+1}} = \ ?$$

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$$\downarrow$$

$$\frac{d_{n,n+1}}{\alpha_{n+1}} \leq \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\sum_{k=1}^{n} \alpha_k(1-\alpha_k)}} \quad ?$$

Upper estimate: $d_{mn} \leq c_{mn}$

Consider the non-optimal transport plan

$$z_{ji} = \begin{cases} \pi_i^n & \text{for } j = i \leq m \\ \pi_j^m \pi_i^n & \text{for } i = m+1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

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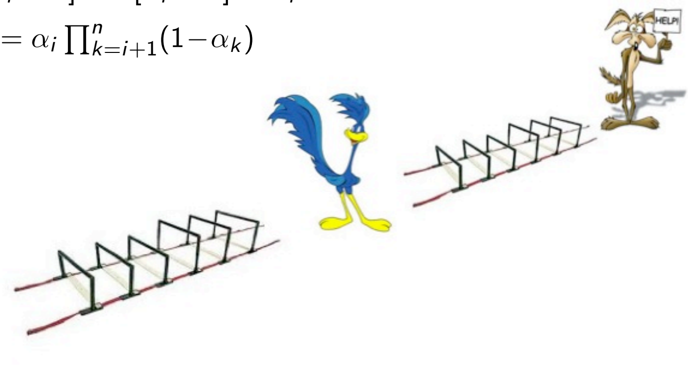
Setting $c_{\text{-}1,n} = 1, c_{\text{-}1,\text{-}1} = 0$, we get inductively

$$\|x^m - x^n\| \leq d_{mn} \leq c_{mn} \triangleq \sum_{j=0}^{m} \sum_{i=m+1}^{n} \pi_j^m \pi_i^n c_{j-1,i-1}$$

Probabilistic interpretation of the recursion

$\mathbb{P}[C_i = 1] = \mathbb{P}[R_i = 1] = \alpha_i$

$\pi_i^n = \alpha_i \prod_{k=i+1}^{n}(1-\alpha_k)$



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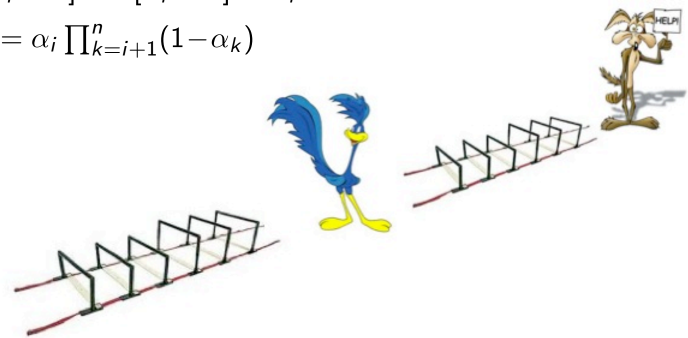

$c_{mn} = \mathbb{P}[\text{roadrunner escapes}]$

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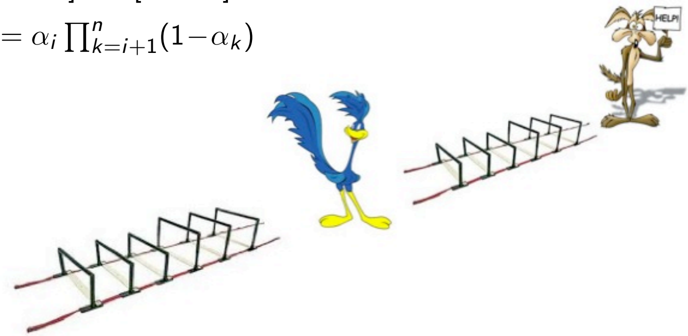

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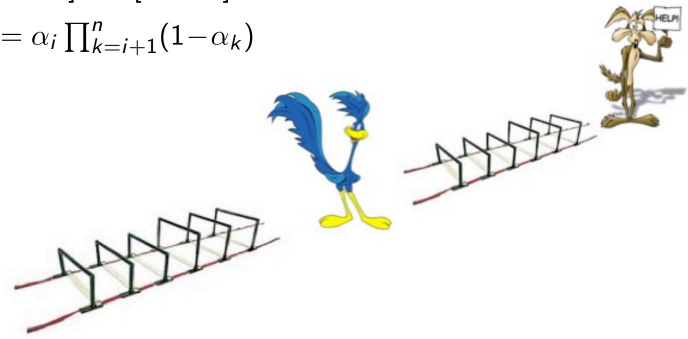

$c_{mn} = \mathbb{P}[\sum_k^n C_i > \sum_k^m R_i, \forall k = m+1, \ldots, 1]$

Coyote must fall more often than Roadrunner

The random walk and the gambler's ruin appear...

$$c_{n,n+1} \quad = \quad \mathbb{P}[\sum_k^{n+1} C_i > \sum_k^n R_i, \forall k = n+1, \ldots, 1]$$

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$$\begin{aligned} c_{n,n+1} &= \mathbb{P}[\textstyle\sum_k^{n+1} C_i > \sum_k^n R_i, \forall k = n+1, \ldots, 1] \\ &= \alpha_{n+1}\, \mathbb{P}[\textstyle\sum_k^n Z_i \geq 0, \forall k = n, \ldots, 1] \end{aligned}$$

$$Z_i = C_i - R_i = \begin{cases} -1 & \text{pbb} \quad \alpha_i(1-\alpha_i) \\ 0 & \text{pbb} \quad 1-2\alpha_i(1-\alpha_i) \\ 1 & \text{pbb} \quad \alpha_i(1-\alpha_i) \end{cases}$$

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$\Rightarrow$ random walk on $\mathbb{Z}$ that moves with probability $p_i = 2\alpha_i(1-\alpha_i)$ and then tosses a coin to decide whether to go left or right

$$\|Tx^n - x^n\| \leq \frac{c_{n,n+1}}{\alpha_{n+1}} = \mathbb{P}[\text{process} \geq 0 \text{ over } n \text{ stages}]$$

An explicit formula for the bound

Rewrite $Z_i = M_i D_i$ with M_i=*move/stay* and D_i=*direction*

$$M_i = \begin{cases} 1 & \text{pbb} \quad p_i \\ 0 & \text{pbb} \quad 1-p_i \end{cases} \quad ; \quad D_i = \begin{cases} -1 & \text{pbb} \quad \frac{1}{2} \\ 1 & \text{pbb} \quad \frac{1}{2} \end{cases}$$

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Conditional on the number of moves $M = M_1 + \ldots + M_n = m$, this is a standard random walk on m stages. The probability for the latter to remain non-negative is $F(m) = \binom{m}{\lfloor m/2 \rfloor} 2^{-m}$, therefore

$$\|x_n - Tx_n\| \leq \frac{c_{n,n+1}}{\alpha_{n+1}} = \sum_{m=0}^{n} F(m)\mathbb{P}[M = m] = \mathbb{E}[F(M)]$$

Sharp bound

Thus (BB) has been reduced to

$$\mathbb{E}[F(M)] \leq \frac{1}{\sqrt{\pi \sum_{i=1}^{n} \alpha_i(1-\alpha_i)}}$$

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$$\sqrt{\tfrac{\pi}{2}(p_1 + \ldots + p_n)}\, \mathbb{E}[F(M_1 + \ldots + M_n)] \leq 1$$

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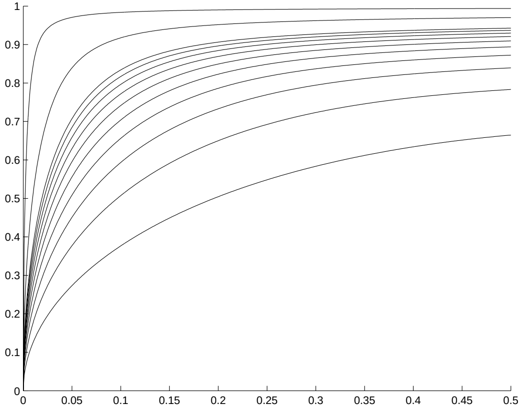
Lemma

$R(p)$ is maximal when $p_i \in \{u, \frac{1}{2}\}$ for some $0 < u < \frac{1}{2}$

Sharp bound: all $p_i = u$

$$R(p) = \sqrt{\tfrac{\pi}{2} n u}\, \mathbb{E}[F(B(n,u))] = \sqrt{\tfrac{\pi}{2} n u}\, {}_2F_1(-n, \tfrac{1}{2}; 2; 2u)$$



Sharp bound: some $p_i = \frac{1}{2}$

Suppose $p_1 = \frac{1}{2}$ and let $S = M_2 + \ldots + M_n$. Conditioning on M_1

$$\mathbb{E}[F(M)] = \mathbb{E}[G(S)]$$

where $G(k) = \frac{1}{2}[F(k) + F(k+1)]$.

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This G is convex so we may use the following Hoeffding-type inequality

Theorem (C-Soto-Vaisman'2014)

Let Z be Poisson with $z = \mathbb{E}(Z) = \mathbb{E}(S)$. Then $\mathbb{E}[G(S)] \leq \mathbb{E}[G(Z)]$.

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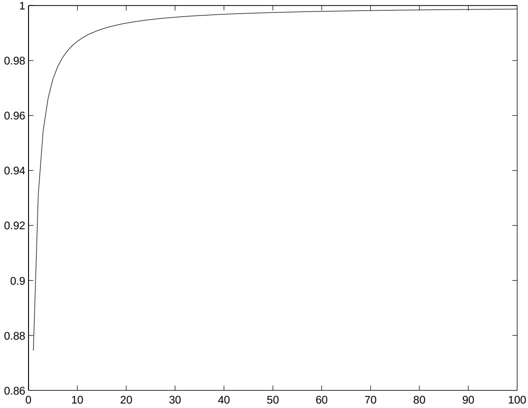
$$\Rightarrow \mathbb{E}[F(M)] \leq \mathbb{E}[G(Z)] = I_0(z) + (1 - \tfrac{1}{2z}) I_1(z)$$

with $I_0(z), I_1(z)$ modified Bessel functions

Sharp explicit bound: some $p_i = \frac{1}{2}$

$$R(p) \leq \sqrt{\tfrac{\pi}{2}\left(\tfrac{1}{2}+z\right)}\,\left[I_0(z) + \left(1-\tfrac{1}{2z}\right)I_1(z)\right]$$



Asymptotic regularity for nonlinear maps

Combining the previous estimates we finally get

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with $\kappa = 1/\sqrt{\pi} \sim 0.5642$

The constant $1/\sqrt{\pi}$ is optimal

Reconsider the optimal recursive bounds

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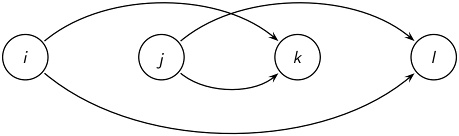
Theorem (Aygen-Satik'2004)

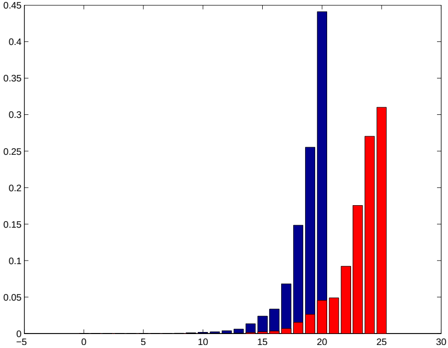
The recursion (R) defines a metric on the set $\{-1,0,1,2,3,\ldots\}$. Moreover, for $i<j<k<l$ we have $d_{ik}+d_{jl}\geq d_{il}+d_{jk}$.



Original proof 80+ pages. Short proof by Bravo-C. (Nov. 2014).

No flow-crossing in optimal transports

$$d_{ik} + d_{jl} \geq d_{il} + d_{jk}$$



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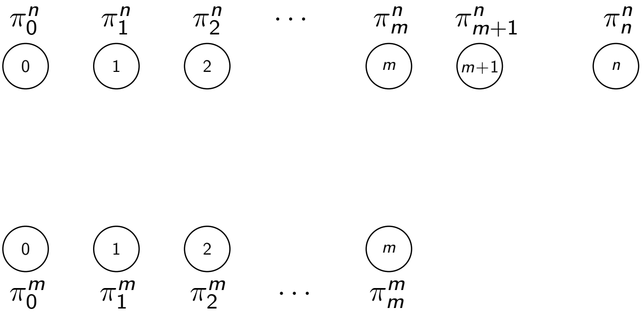

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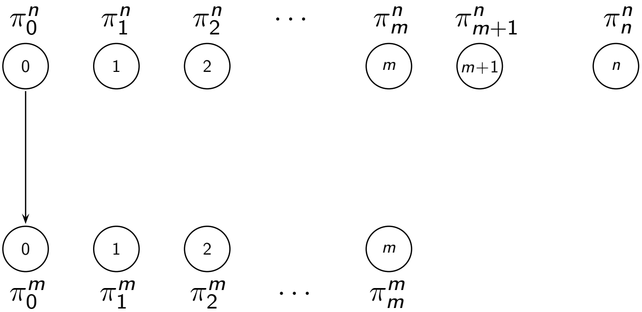

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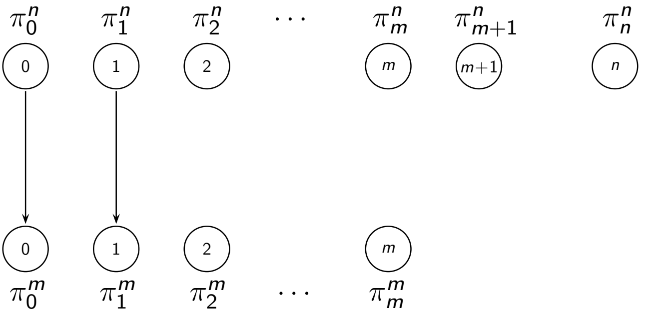

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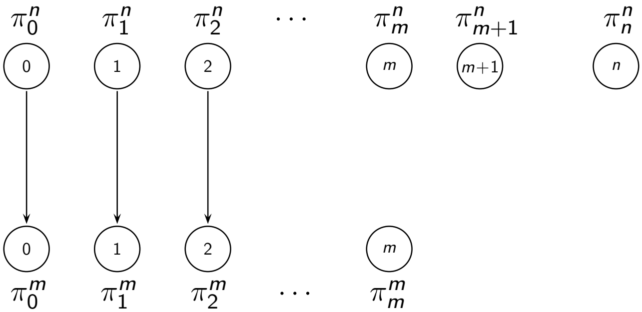

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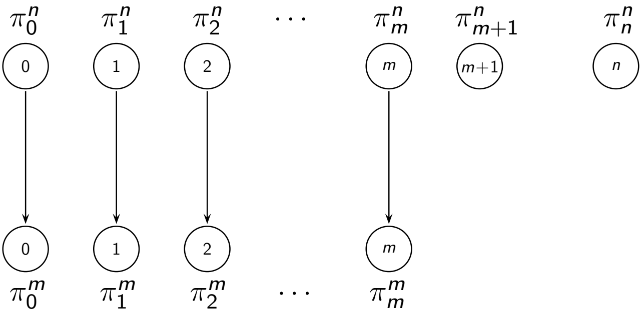

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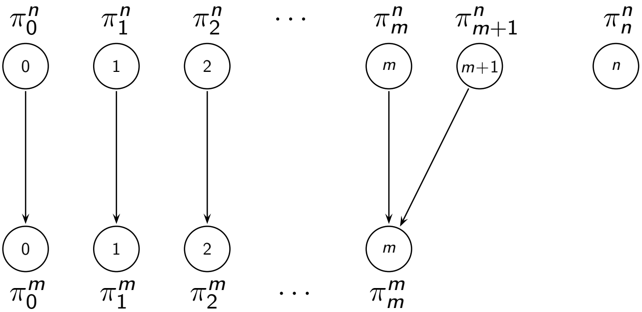

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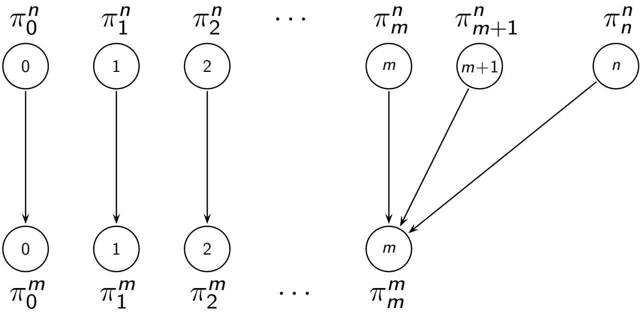

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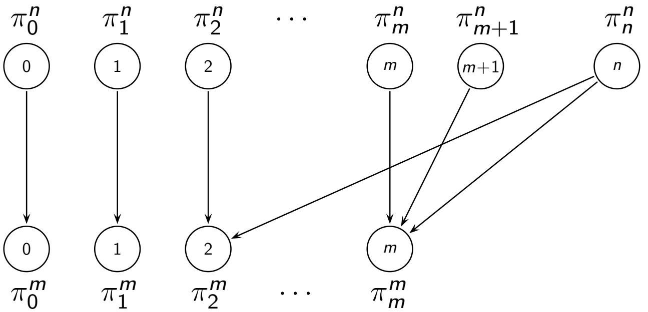

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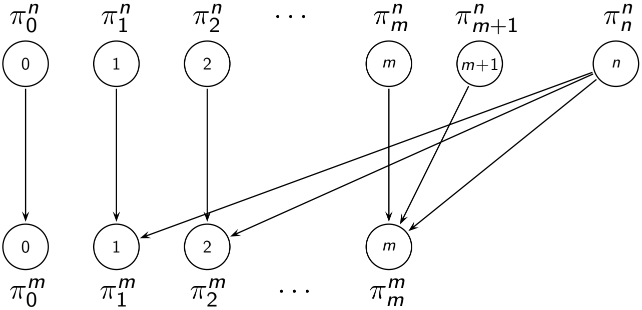

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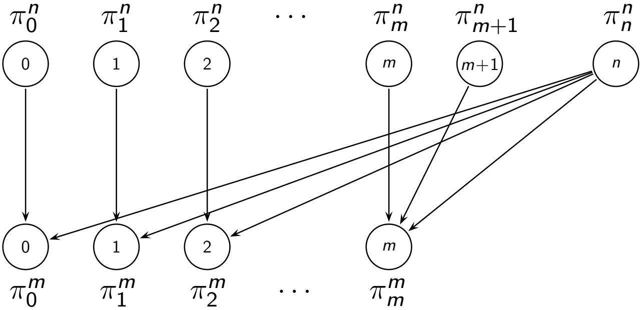
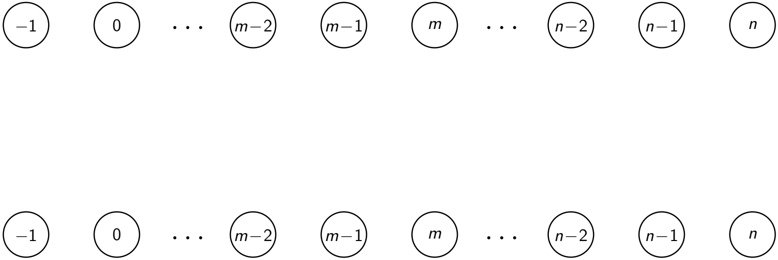


Stochastic process for d_{mn} (case $\alpha_n \equiv \alpha \geq \frac{1}{2}$)



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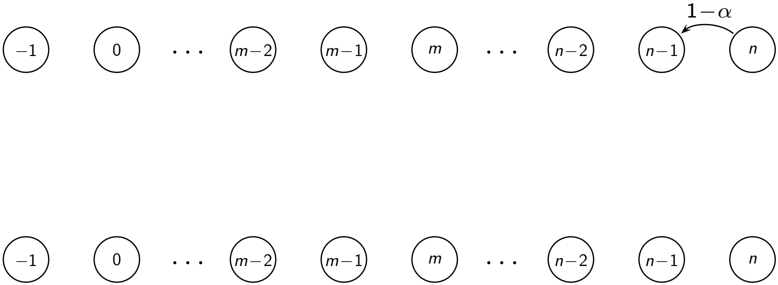

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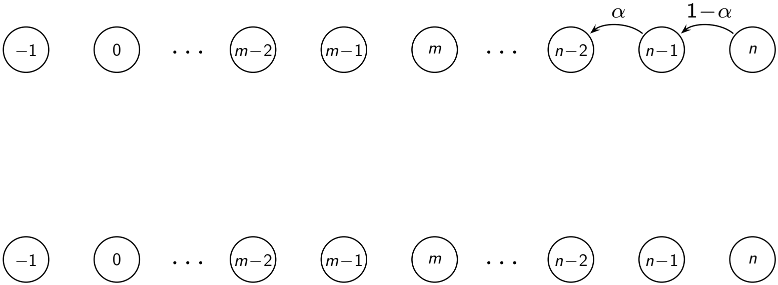

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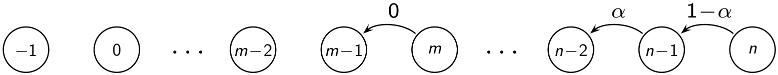

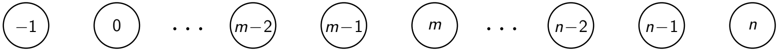

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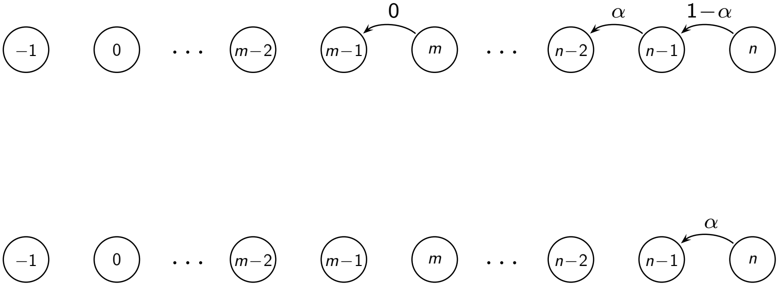

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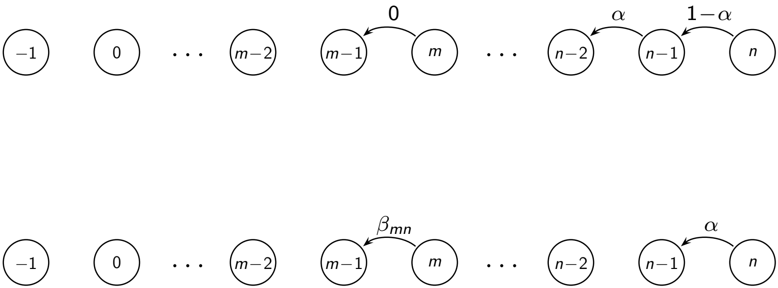

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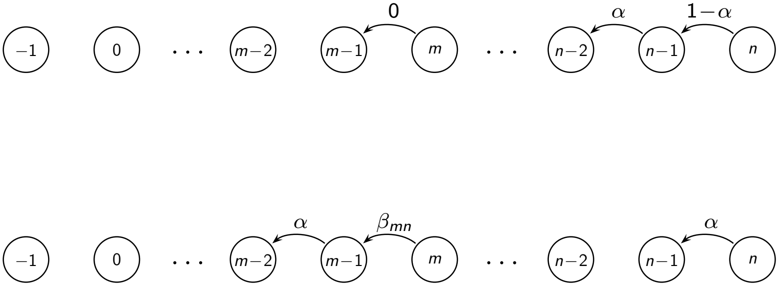

The constant $\frac{1}{\sqrt{\pi}}$ is optimal

Transition probabilities

Process d_{mn}: $P_{mn}^{ij} = \mathbb{P}((m,n) \searrow (i,j))$

Process c_{mn}: $Q_{mn}^{ij} = \mathbb{P}((m,n) \searrow (i,j))$

We have $\|P - Q\| \leq 4(1-\alpha)^2$ for the ℓ^∞ operator norm $\|\cdot\|$.

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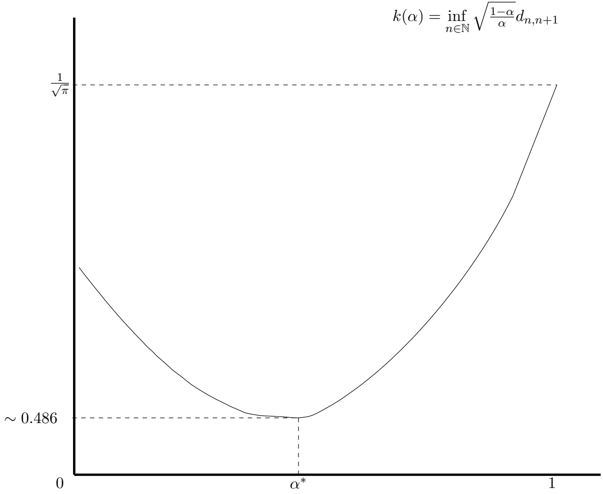
Taking $\alpha_n = 1 - \log n/n$ we get

$$\lim_n \sqrt{n\alpha_n(1-\alpha_n)}\,\frac{d_{n,n+1}}{\alpha_n} = \lim_n \sqrt{n\alpha_n(1-\alpha_n)}\,\frac{c_{n,n+1}}{\alpha_n} = \frac{1}{\sqrt{\pi}}$$

Theorem (Bravo-C., April 2015)

The constant $\frac{1}{\sqrt{\pi}}$ in (BB) is sharp.

Optimal constant for fixed α ?



$\alpha^* \sim 0.46 < \frac{1}{2}$?

$$\|Tx^n - x^n\| \leq \frac{1}{\sqrt{\pi}} \frac{\operatorname{diam}(C)}{\sqrt{\sum_{k=1}^{n} \alpha_k(1-\alpha_k)}}$$

Feliz cumpleaños Terry !

Affine Maps

Affine maps

Example: Right-shift on $\ell^1(\mathbb{N})$

$C = \{p \in \ell^1(\mathbb{N}) : p_i \geq 0, \sum_{i=0}^{\infty} p_i = 1\}$ with $\mathrm{diam}(C) = 2$

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\begin{aligned}
x^0 &= (1, 0, 0, 0, \ldots) \\
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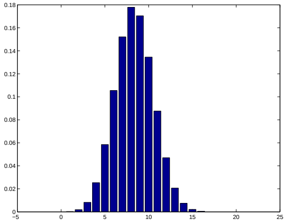

$x_k^n = \mathbb{P}(X_1 + \cdots + X_n = k)$

$X_i \sim$Bernoulli(α_i)

$\|Tx^n - x^n\|_1 = 2\max_k x_k^n$

Sums of Bernoullis and (*BB*)

Theorem (Baillon-C-Vaisman, arXiv'2013)

Let X_i be independent Bernoullis with $\mathbb{P}(X_i\!=\!1) = \alpha_i$. Then

$$p_k^n = \mathbb{P}(X_1 + \ldots + X_n = k) \leq \frac{\eta}{\sqrt{\sum_{i=1}^n \alpha_i(1-\alpha_i)}}$$

where $\eta = \max_{u\geq 0} \sqrt{u}\, e^{-u} I_0(u) \sim 0.4688$ *with* $I_0(\cdot)$ *modified Bessel function. This bound is sharp.*

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Corollary

For the right shift in $\ell^1(\mathbb{N})$ the optimal bound in (BB) is $\kappa = \eta$.

Asymptotic regularity for affine maps

Let $\bar{x} \in \mathrm{Fix}\, T$ and $C = B(\bar{x}, r)$ with $r = \|x^0 - \bar{x}\|$ so that $T : C \to C$.

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Corollary

For affine maps (BB) holds with $\kappa = \eta$. This bound is sharp and is attained by the right shift.