

Fixed-Point Iterations for Nonexpansive Maps

The sharp bound of asymptotic regularity is $1/\sqrt{\pi}$

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Based on joint work with
J-B. Baillon, M. Bravo, J. Soto, J. Vaisman

CELEBRATING TERRY ROCKAFELLAR'S 80TH BIRTHDAY

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Banach-Picard fixed point iteration

$T : X \rightarrow X$ contraction

(BP)

$$x^{n+1} = Tx^n$$

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$$\|x^{n+1} - x^n\| = \|Tx^n - x^n\| \leq \rho^n \|Tx^0 - x^0\| \rightarrow 0$$

↓

convergence + error estimates + stopping rule

Krasnoselskii-Mann fixed point iteration

$T : C \rightarrow C$ non-expansive / C convex bounded in $(X, \|\cdot\|)$

(KM)

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- technique for proving existence of fixed points
- convex optimization: Prox, Douglas-Rachford, Peaceman-Rachford...
- discretization of semigroups $\frac{dx}{dt} + [I - T](x) = 0$
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Question: $\|Tx^n - x^n\| \rightarrow 0$? (Asymptotic Regularity)

History of $\|Tx^n - x^n\| \rightarrow 0$

- 1955 **Krasnoselskii**: X uniformly convex, C compact, $\alpha_n \equiv 1/2$
- 1957 **Schaefer**: extends Krasnoselskii to $\alpha_n \equiv \alpha$
- 1966 **Ishikawa**: extends Krasnoselskii to X strictly convex
- 1966 **Browder-Petryshyn**: X uniformly convex, $\text{Fix}(T) \neq \emptyset$, $\alpha_n \equiv \alpha$
- 1976 **Ishikawa**: X general, C compact, $\sum \alpha_n = \infty$, $\alpha_n \leq 1 - \epsilon$
- 1978 **Edelstein-O'Brien**: $\|Tx^n - x^n\| \rightarrow 0$ uniformly w.r.t x_0
- 1983 **Goebel-Kirk**: uniformly w.r.t. x_0 and T (for C given)
- 1992 **Baillon-Bruck**: $\alpha_n \equiv \alpha \Rightarrow \|Tx^n - x^n\| = O(1/\log n)$
- 2003 **Kohlenbach**: $\|Tx^n - x^n\| \rightarrow 0$ only depends on $\text{diam}(C)$

Baillon-Bruck's conjecture (1992)

There exists a universal constant κ such that

$$(BB) \quad \|Tx^n - x^n\| \leq \kappa \frac{\text{diam}(C)}{\sqrt{\sum_{k=1}^n \alpha_k (1 - \alpha_k)}}$$

REMARK: in continuous time $\|Tx(t) - x(t)\| \leq \kappa \frac{\text{diam}(C)}{\sqrt{t}}$

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Theorem (Baillon-Bruck'1996)

When $\alpha_n \equiv \alpha$ the bound holds with $\kappa = 1/\sqrt{\pi}$.

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Our contributions:

- **The bound holds for general α_n with $\kappa = 1/\sqrt{\pi} \sim 0.5642$**
- **Nonlinear maps: the constant $\kappa = 1/\sqrt{\pi}$ is optimal**
- **Affine maps: sharp bound with $\kappa = \max_z \sqrt{ze^{-z}} I_0(z) \sim 0.4688$**

Nonlinear Maps

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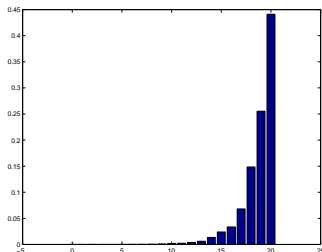
Recall

$$x^{n+1} = (1 - \alpha_{n+1})x^n + \alpha_{n+1}Tx^n$$

Alternative expression for x^n

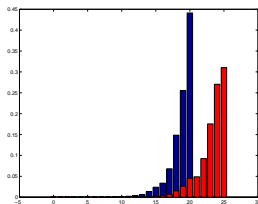
Let $\pi_i^n = \alpha_i \prod_{k=i+1}^n (1 - \alpha_k)$ and set $Tx^{-1} = x_0$ by convention, then

$$x^n = \sum_{i=0}^n \pi_i^n Tx^{i-1}$$



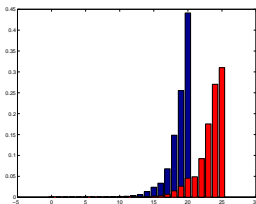
A recursive bound $\|x^m - x^n\| \leq d_{mn}$

$$x^m - x^n = \sum_{j=0}^m \pi_j^m T x^{j-1} - \sum_{i=0}^n \pi_i^n T x^{i-1}$$



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$$x^m - x^n = \sum_{j=0}^m \pi_j^m T_X^{j-1} - \sum_{i=0}^n \pi_i^n T_X^{i-1}$$



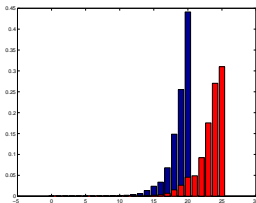
Let P_{mn} be the set of transport plans $z \geq 0$ taking π^n to π^m

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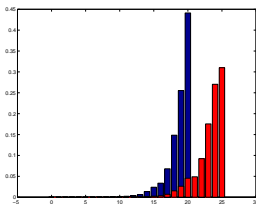


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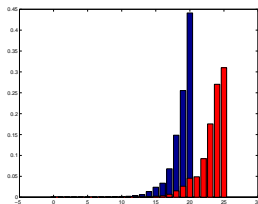
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$$\|x^m - x^n\| \leq \sum_{j=0}^m \sum_{i=0}^n z_{ji} \|x^{j-1} - x^{i-1}\|$$

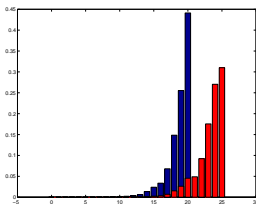


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$$\|x^m - x^n\| \leq \sum_{j=0}^m \sum_{i=0}^n z_{ji} d_{j-1, i-1} \quad \leftarrow \quad \min_z$$



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Set $d_{-1,n} = 1$, $d_{-1,-1} = 0$, and define inductively

$$(R) \quad d_{mn} \triangleq \min_{z \in P_{mn}} \sum_{j=0}^m \sum_{i=0}^n z_{ji} d_{j-1,i-1}$$

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Theorem (Bravo-C. – Sep. 2014)

Bound (R) is best possible: There exists a non-expansive T on the set $C = [0, 1]^{\mathbb{N}} \subseteq \ell^\infty(\mathbb{N})$ which attains $\|x^m - x^n\| = d_{mn}$ for all m, n .

Proof: Built from dual solutions of the optimal transports.

Restatement of (BB)

$$\|Tx^n - x^n\| = \left\| \frac{x^{n+1} - x^n}{\alpha_{n+1}} \right\| \leq \frac{d_{n,n+1}}{\alpha_{n+1}} = ?$$

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$$\frac{d_{n,n+1}}{\alpha_{n+1}} \leq \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\sum_{k=1}^n \alpha_k (1 - \alpha_k)}} \quad ?$$

Upper estimate: $d_{mn} \leq c_{mn}$

Consider the non-optimal transport plan

$$z_{ji} = \begin{cases} \pi_i^n & \text{for } j = i \leq m \\ \pi_j^m \pi_i^n & \text{for } i = m + 1, \dots, n \\ 0 & \text{otherwise} \end{cases}$$

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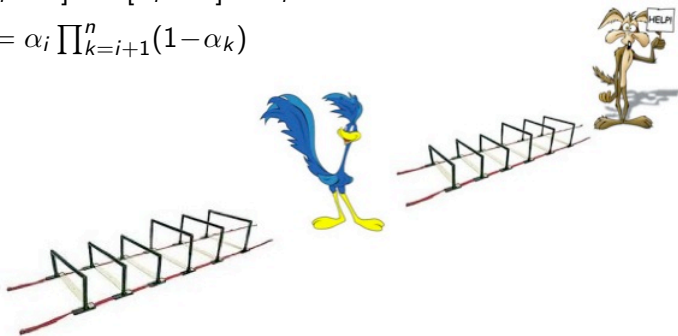
Setting $c_{-1,n} = 1$, $c_{-1,-1} = 0$, we get inductively

$$\|x^m - x^n\| \leq d_{mn} \leq c_{mn} \triangleq \sum_{j=0}^m \sum_{i=m+1}^n \pi_j^m \pi_i^n c_{j-1,i-1}$$

Probabilistic interpretation of the recursion

$$\mathbb{P}[C_i = 1] = \mathbb{P}[R_i = 1] = \alpha_i$$

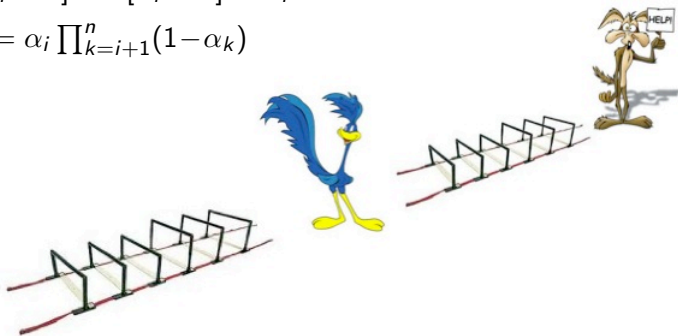
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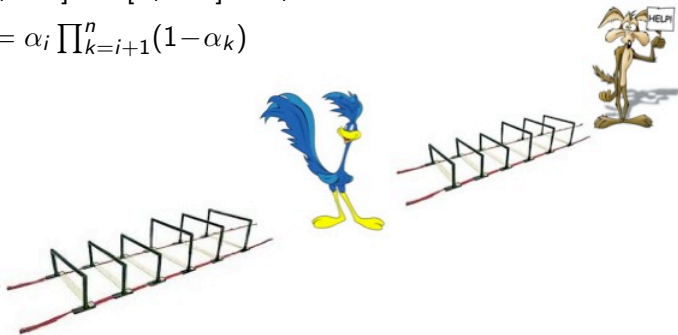


$$c_{mn} = \mathbb{P}[\text{roadrunner escapes}]$$

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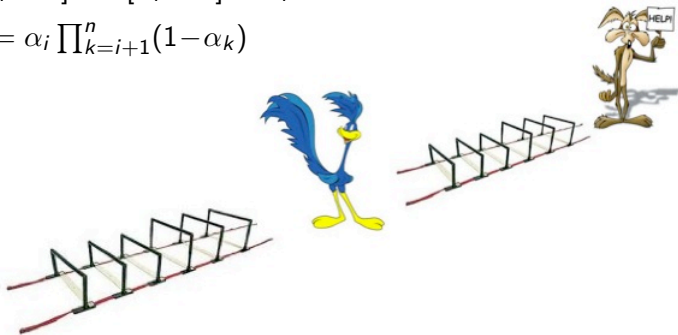


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$$c_{mn} = \mathbb{P}[\sum_k^n C_i > \sum_k^m R_i, \forall k = m + 1, \dots, n]$$

Coyote must fall more often than Roadrunner

The random walk and the gambler's ruin appear...

$$c_{n,n+1} = \mathbb{P}[\sum_k^{n+1} C_i > \sum_k^n R_i, \forall k = n+1, \dots, 1]$$

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$$\begin{aligned}
 c_{n,n+1} &= \mathbb{P}[\sum_k^{n+1} C_i > \sum_k^n R_i, \forall k = n+1, \dots, 1] \\
 &= \alpha_{n+1} \mathbb{P}[\sum_k^n Z_i \geq 0, \forall k = n, \dots, 1]
 \end{aligned}$$

$$Z_i = C_i - R_i = \begin{cases} -1 & \text{pbb} & \alpha_i(1 - \alpha_i) \\ 0 & \text{pbb} & 1 - 2\alpha_i(1 - \alpha_i) \\ 1 & \text{pbb} & \alpha_i(1 - \alpha_i) \end{cases}$$

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⇒ random walk on \mathbb{Z} that moves with probability $p_i = 2\alpha_i(1 - \alpha_i)$ and then tosses a coin to decide whether to go left or right

$$\|T_X^n - x^n\| \leq \frac{c_{n,n+1}}{\alpha_{n+1}} = \mathbb{P}[\text{process} \geq 0 \text{ over } n \text{ stages}]$$

An explicit formula for the bound

Rewrite $Z_i = M_i D_i$ with $M_i = \text{move/stay}$ and $D_i = \text{direction}$

$$M_i = \begin{cases} 1 & \text{pbb} & p_i \\ 0 & \text{pbb} & 1 - p_i \end{cases} \quad ; \quad D_i = \begin{cases} -1 & \text{pbb} & \frac{1}{2} \\ 1 & \text{pbb} & \frac{1}{2} \end{cases}$$

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Conditional on the number of moves $M = M_1 + \dots + M_n = m$, this is a standard random walk on m stages. The probability for the latter to remain non-negative is $F(m) = \binom{m}{\lfloor m/2 \rfloor} 2^{-m}$, therefore

$$\|x_n - Tx_n\| \leq \frac{c_{n,n+1}}{\alpha_{n+1}} = \sum_{m=0}^n F(m) \mathbb{P}[M = m] = \mathbb{E}[F(M)]$$

Sharp bound

Thus (BB) has been reduced to

$$\mathbb{E}[F(M)] \leq \frac{1}{\sqrt{\pi \sum_{i=1}^n \alpha_i (1 - \alpha_i)}}$$

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Since $p_i = 2\alpha_i(1 - \alpha_i)$ this is equivalent to

$$\sqrt{\frac{\pi}{2}(p_1 + \dots + p_n)} \mathbb{E}[F(M_1 + \dots + M_n)] \leq 1$$

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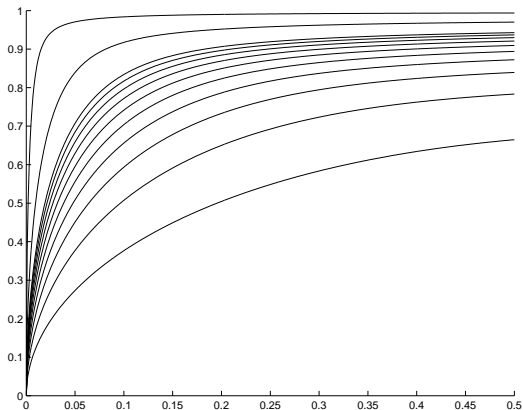
$$\underbrace{\sqrt{\frac{\pi}{2}(p_1 + \dots + p_n)} \mathbb{E}[F(M_1 + \dots + M_n)]}_{R(p)} \leq 1$$

Lemma

$R(p)$ is maximal when $p_i \in \{u, \frac{1}{2}\}$ for some $0 < u < \frac{1}{2}$

Sharp bound: all $p_i = u$

$$R(p) = \sqrt{\frac{\pi}{2} nu} \mathbb{E}[F(B(n, u))] = \sqrt{\frac{\pi}{2} nu} {}_2F_1\left(-n, \frac{1}{2}; 2; 2u\right)$$



Sharp bound: some $p_i = \frac{1}{2}$

Suppose $p_1 = \frac{1}{2}$ and let $S = M_2 + \dots + M_n$. Conditioning on M_1

$$\mathbb{E}[F(M)] = \mathbb{E}[G(S)]$$

where $G(k) = \frac{1}{2}[F(k) + F(k + 1)]$.

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This G is convex so we may use the following Hoeffding-type inequality

Theorem (C-Soto-Vaisman'2014)

Let Z be Poisson with $z = \mathbb{E}(Z) = \mathbb{E}(S)$. Then $\mathbb{E}[G(S)] \leq \mathbb{E}[G(Z)]$.

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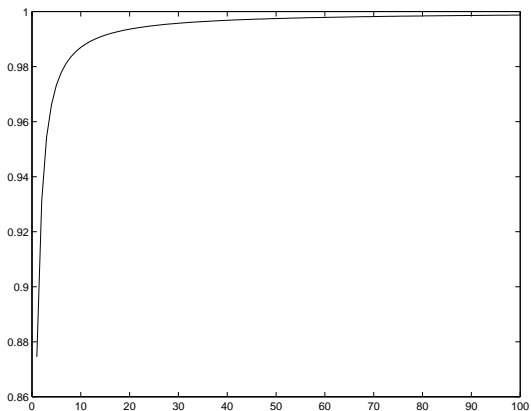
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$$\Rightarrow \mathbb{E}[F(M)] \leq \mathbb{E}[G(Z)] = l_0(z) + (1 - \frac{1}{2z})l_1(z)$$

with $l_0(z), l_1(z)$ modified Bessel functions

Sharp explicit bound: some $p_i = \frac{1}{2}$

$$R(p) \leq \sqrt{\frac{\pi}{2} \left(\frac{1}{2} + z \right) \left[l_0(z) + \left(1 - \frac{1}{2z} \right) l_1(z) \right]}$$



Asymptotic regularity for nonlinear maps

Combining the previous estimates we finally get

Theorem (C-Soto-Vaisman'2014)

$$(BB) \quad \|Tx^n - x^n\| \leq \kappa \frac{\text{diam}(C)}{\sqrt{\sum_{k=1}^n \alpha_k (1 - \alpha_k)}}$$

with $\kappa = 1/\sqrt{\pi} \sim 0.5642$

The constant $1/\sqrt{\pi}$ is optimal

Reconsider the optimal recursive bounds

$$(R) \quad d_{mn} \triangleq \min_{z \in P_{mn}} \sum_{j=0}^m \sum_{i=0}^n z_{ji} d_{j-1, i-1}$$

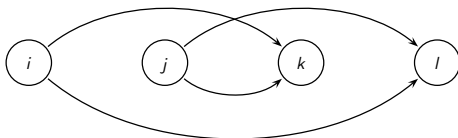
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Theorem (Aygen-Satik'2004)

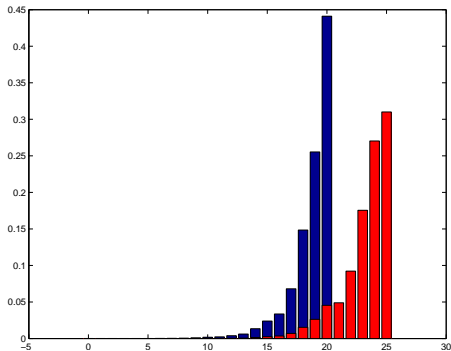
*The recursion (R) defines a metric on the set $\{-1, 0, 1, 2, 3, \dots\}$.
Moreover, for $i < j < k < l$ we have $d_{ik} + d_{jl} \geq d_{il} + d_{jk}$.*



Original proof 80+ pages. Short proof by Bravo-C. (Nov. 2014).

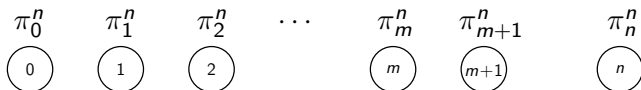
No flow-crossing in optimal transports

$$d_{ik} + d_{jl} \geq d_{il} + d_{jk}$$



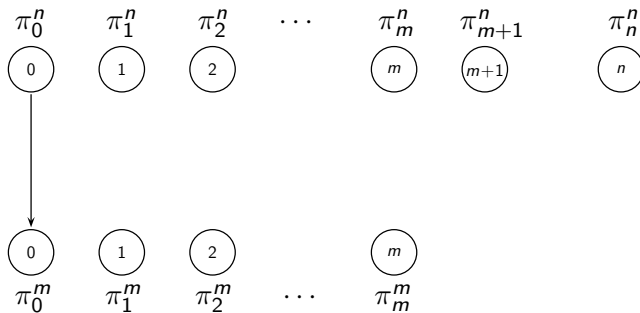
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For $\alpha_n \equiv \alpha \geq \frac{1}{2}$ the optimal transport is



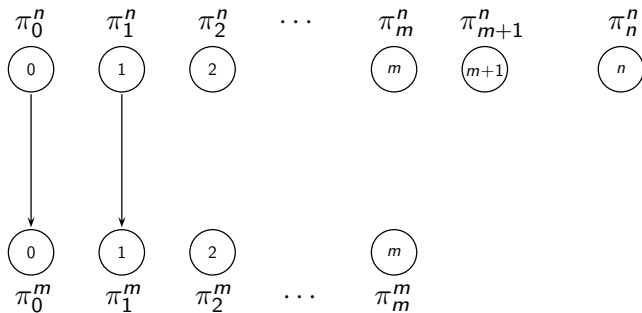
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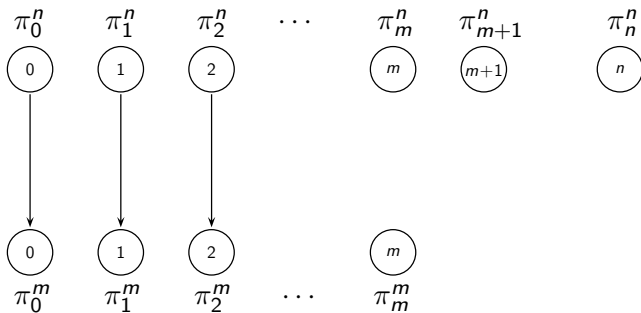
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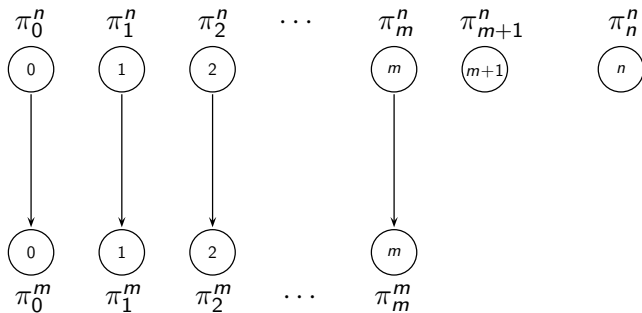
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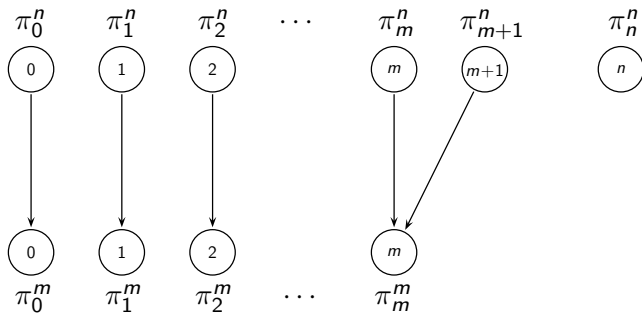
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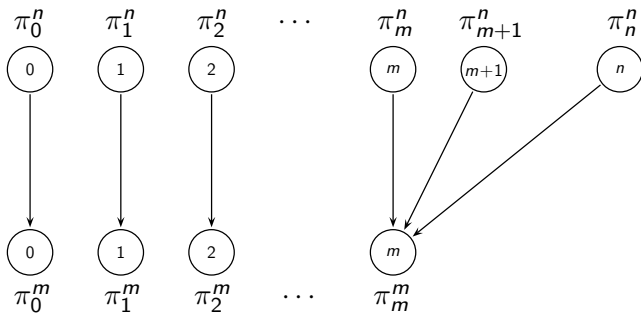
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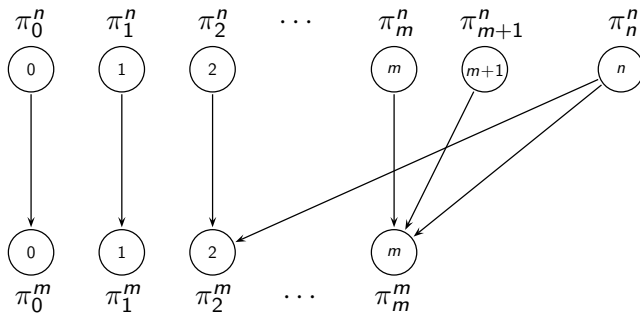
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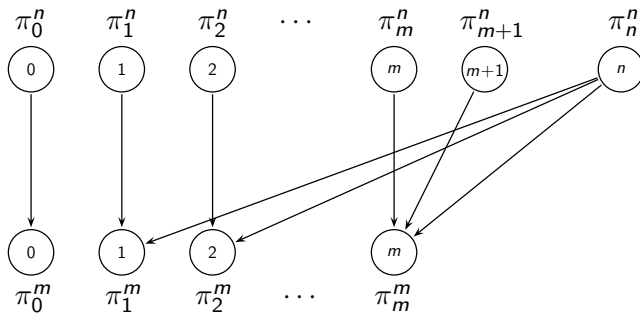
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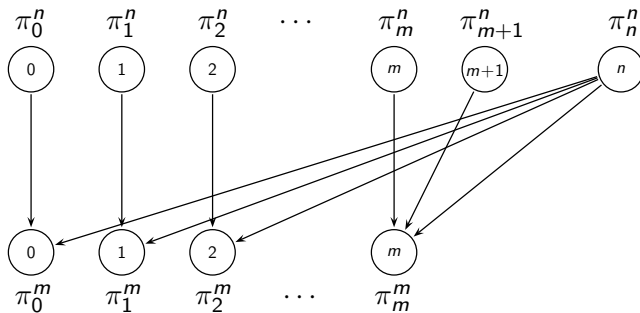
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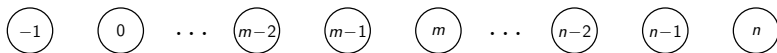


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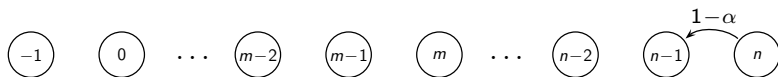
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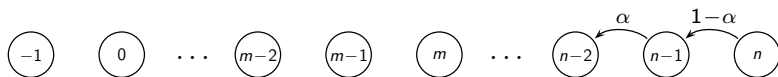
Stochastic process for d_{mn} (case $\alpha_n \equiv \alpha \geq \frac{1}{2}$)



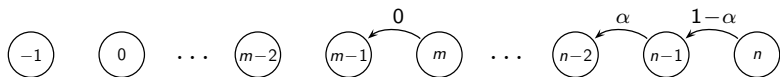
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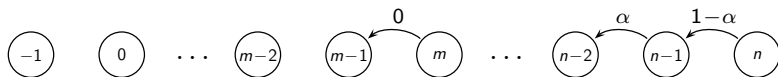
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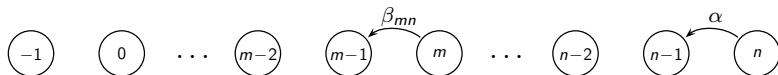
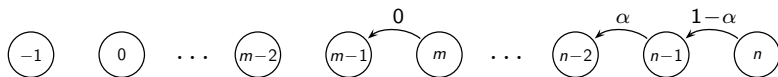
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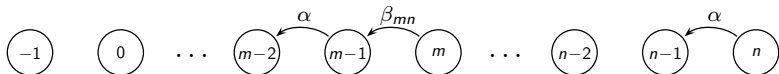
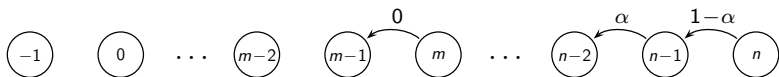
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The constant $\frac{1}{\sqrt{\pi}}$ is optimal

Transition probabilities

$$\text{Process } d_{mn}: P_{mn}^{ij} = \mathbb{P}((m, n) \searrow (i, j))$$

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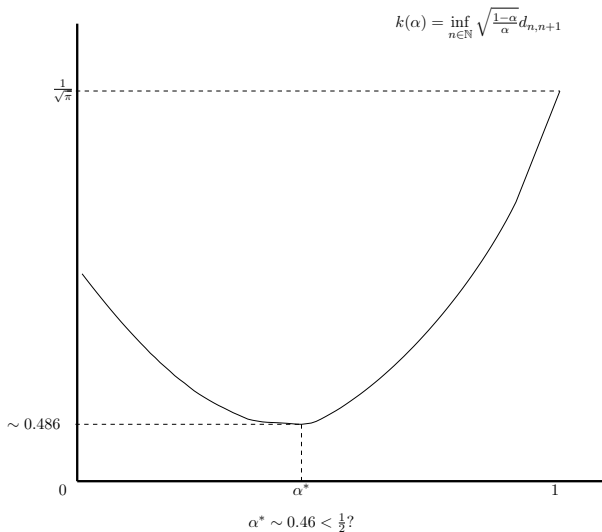
Taking $\alpha_n = 1 - \log n/n$ we get

$$\lim_n \sqrt{n\alpha_n(1-\alpha_n)} \frac{d_{n,n+1}}{\alpha_n} = \lim_n \sqrt{n\alpha_n(1-\alpha_n)} \frac{c_{n,n+1}}{\alpha_n} = \frac{1}{\sqrt{\pi}}$$

Theorem (Bravo-C., April 2015)

The constant $\frac{1}{\sqrt{\pi}}$ in (BB) is sharp.

Optimal constant for fixed α ?



$$\|Tx^n - x^n\| \leq \frac{1}{\sqrt{\pi}} \frac{\text{diam}(C)}{\sqrt{\sum_{k=1}^n \alpha_k(1 - \alpha_k)}}$$

Feliz cumpleaños Terry !

Affine Maps

Affine maps

Example: Right-shift on $\ell^1(\mathbb{N})$

$C = \{p \in \ell^1(\mathbb{N}) : p_i \geq 0, \sum_{i=0}^{\infty} p_i = 1\}$ with $\text{diam}(C) = 2$

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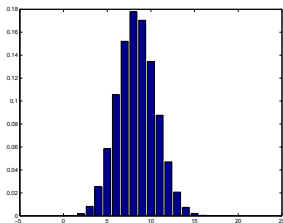
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$$X_i \sim \text{Bernoulli}(\alpha_i)$$

$$\|Tx^n - x^n\|_1 = 2 \max_k x_k^n$$

Sums of Bernoullis and (BB)

Theorem (Baillon-C-Vaisman, arXiv'2013)

Let X_i be independent Bernoullis with $\mathbb{P}(X_i=1) = \alpha_i$. Then

$$p_k^n = \mathbb{P}(X_1 + \dots + X_n = k) \leq \frac{\eta}{\sqrt{\sum_{i=1}^n \alpha_i(1 - \alpha_i)}}$$

where $\eta = \max_{u \geq 0} \sqrt{u} e^{-u} I_0(u) \sim 0.4688$ with $I_0(\cdot)$ modified Bessel function. This bound is sharp.

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Corollary

For the right shift in $\ell^1(\mathbb{N})$ the optimal bound in (BB) is $\kappa = \eta$.

Asymptotic regularity for affine maps

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Corollary

For affine maps (BB) holds with $\kappa = \eta$. This bound is sharp and is attained by the right shift.