Three integration theorems of the Fenchel subdifferential of nonconvex functions[1]

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[1] Joint work with Y. Garcia and A. Hantoute

Subdifferential of the inf-convolution

- Our framework is a pair of lacally convex spaces paired in duality, (X, X^*)
- Our functions are extended-valued non-necessarily convex
- The (approximate) subdifferential is the classical subdifferential of convex analysis

We start by establishing approximate subdifferential calculus formulas for the inf-convolution, the conjugate function, the closure, the convex envelope, the closed convex envelope

Proposition : Let $f, g : X \to \overline{\mathbb{R}}$ be two proper functions. Then for every x and $\varepsilon \geq 0$ we have that

$$\partial_{\varepsilon}(f \square g)(x) = \bigcap_{\alpha > 0} \bigcup_{\substack{y \in X \\ \varepsilon_1, \varepsilon_2 > 0,\ \varepsilon_1 + \varepsilon_2 \leq \varepsilon + \alpha}} \partial_{\varepsilon_1} f(y) \cap \partial_{\varepsilon_2} g(x - y)$$

Subdifferential of the closure function

Given $f : X \to \overline{\mathbb{R}}$ with proper conjugate, for every $x \in X$ and $\varepsilon \geq 0$,

$$\partial_\varepsilon (\operatorname{cl} f)(x) = \bigcap_{\substack{\delta > 0 \\ V \in \mathcal{V}_x}} \bigcup_{y \in V} \partial_{\varepsilon + \delta} f(y)$$

Equivalently,

$$\left(\partial_\varepsilon (\operatorname{cl} f)\right)^{-1}(x^*) = \bigcap_{\delta > 0} \overline{\left(\partial_{\varepsilon+\delta} f\right)^{-1}(x^*)}$$

Consequently, if $f : X \to \overline{\mathbb{R}}$ is convex,

$$\partial_\varepsilon f^*(x^*) = \bigcap_{\delta > 0} \overline{(\partial_{\varepsilon+\delta} f)^{-1}(x^*)}$$

This should be compared with the Fenchel duality in $\Gamma_0(X)$

$$\partial_\varepsilon f^*(x^*) = (\partial_\varepsilon f)^{-1}(x^*)$$

Subdifferential of the convex envelope

Given $f : X \to \overline{\mathbb{R}}$, for every $x \in X$ and $\varepsilon \geq 0$ we have

$$\partial_\varepsilon (\operatorname{co} f)(x) = \bigcap_{\delta > 0} \bigcup_{\substack{k \in \mathbb{N} \\ (\varepsilon_1, \ldots, \varepsilon_k, x_1, \ldots, x_k) \in \Delta_k(x, \varepsilon + \delta)}} \bigcap_{i = 1, \ldots, k} \partial_{\varepsilon_i} f(x_i)$$

where

$$\Delta_k(x, \alpha) := \Big\{ (\varepsilon_1, \cdots, \varepsilon_k, x_1, \cdots, x_k) \mid \varepsilon_i \geq 0,\ x_i \in X,\ \exists\ (\lambda_1, \cdots, \lambda_k) \in \Delta_k \\ \text{s.t.} \sum_{i=1,\cdots,k} \lambda_i \varepsilon_i \leq \alpha, \sum_{i=1,\cdots,k} \lambda_i x_i = x \Big\},\ \alpha \geq 0$$

Recall that (assuming f^*proper)

$$(\operatorname{co} f)(x) = \inf \left\{ \begin{array}{c} \sum_{i=1,\cdots,k} \lambda_i f(x_i) \mid \sum_{i=1,\cdots,k} \lambda_i x_i = x,\ x_i \in \operatorname{dom} f, \\ (\lambda_1, \cdots, \lambda_k) \in \Delta_k,\ k \in \mathbb{N} \end{array} \right\}$$

Subdifferential of the closed convex envelope

For $f : X \to \overline{\mathbb{R}}$ with proper conjugate we have, for every $x \in X$ and $\varepsilon \geq 0$,

$$\partial_\varepsilon(\overline{\text{co}} f)(x) = \bigcap_{\substack{\delta>0 \\ V\in\mathcal{V}_x}} \bigcup_{\substack{y\in V,\ k\in\mathbb{N} \\ (\varepsilon_1,\ldots,\varepsilon_k,y_1,\ldots,y_k)\in\Delta_k(y,\varepsilon+\delta)}} \bigcap_{i=1,\ \ldots,k} \partial_{\varepsilon_i} f(y_i)$$

Consequently, by inverting this formula,

$$\partial_\varepsilon f^*(x^*) = \bigcap_{\delta>0} \text{cl}\left\{ \sum_{i=1,\cdots,k} \lambda_i \partial_{\varepsilon_i} f(y_i) \mid \lambda_i \geq 0,\ \textstyle\sum_{i=1,\cdots,k} \lambda_i \varepsilon_i \leq \varepsilon + \delta,\ k \in \mathbb{N} \right\}$$

Subdifferential of the closed convex envelope - an other formula

Given a function $f : X \to \overline{\mathbb{R}}$, we assume that $\operatorname{ri}(\operatorname{dom} f^*) \neq \emptyset$. Then for every $x \in X$ we have that

$$\partial(\overline{\operatorname{co}} f)(x) = \bigcap_{\varepsilon>0, V\in\mathcal{V}_x} \bigcup_{\substack{(\lambda_1,\cdots,\lambda_k)\in\Delta_k,\ k\in\mathbb{N} \\ \sum_{i=1,\cdots,k} \lambda_i y_i + z \in V,\ z\in X}} \bigcap_{i=1,\ k} \partial_\varepsilon f(y_i) \bigcap \partial\sigma_{\operatorname{dom} f^*}(z)$$

Equivalently, we have that

$$\partial(\overline{\operatorname{co}} f)(x) = \bigcap_{\varepsilon>0, V\in\mathcal{V}_x} \bigcup_{\substack{(\lambda_1,\cdots,\lambda_k)\in\Delta_k,\ k\in\mathbb{N} \\ \sum_{i=1,\cdots,k} \lambda_i y_i + z \in V,\ z\in X}} \bigcap_{i=1,\ k} \partial_\varepsilon f(y_i) \bigcap \partial f^\infty(z)$$

where $f^\infty(x) := \liminf_{s\to 0^+,\ \langle y,\cdot\rangle \to \langle x,\cdot\rangle \text{ on } \operatorname{dom} f^*} s f(s^{-1} y)$

Integration theorems

Given $f, g : X \to \overline{\mathbb{R}}$ with f^* being proper. For $\delta \in (0, \infty]$ we suppose

$$\partial_\varepsilon f(x) \subset \partial_\varepsilon g(x), \quad \text{for all } x \in X \text{ and all } \varepsilon \in (0, \delta)$$

Theorem 1: If δ is finite, then

$$f^* = g^* + \mathrm{I}_{\mathrm{dom}\, f^*} + \mathrm{cte}$$

Theorem 2: If $\delta = \infty$, then

$$\overline{\mathrm{co}} f = \overline{\mathrm{co}} g + \mathrm{cte}$$

Theorem 3: If δ is finite, then

$$\overline{\mathrm{co}} f = \sup_{L \in \mathcal{F}} \overline{(\overline{\mathrm{co}} g) \,\Box\, \sigma_{\mathrm{dom}\, f^* \cap L}} + \mathrm{cte}$$

where $\mathcal{F}$ refers to the family of finite-dimensional subspaces of X^*.
In particular, if $\mathrm{ri}(\mathrm{dom}\, f^*) \neq \emptyset$, or if g^* is continuous at some point in $\mathrm{dom}\, f^*$, or if $\mathrm{dom}\, f^*$ is closed, then

$$\overline{\mathrm{co}} f = \overline{(\overline{\mathrm{co}} g) \,\Box\, \sigma_{\mathrm{dom}\, f^*}} + \mathrm{cte}$$

Example

Consider the functions $f, g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ defined as

$$g(x) := \mathrm{I}_{\{0\}} \text{ and } f(x) := \begin{cases} |x| + 1 & \textit{if } x \neq 0 \\ 0 & \textit{if } x = 0 \end{cases}$$

- $f^* = \mathrm{I}_{[-1,1]} - 1, \;\; g^* \equiv 0$
- $(\operatorname{co} f)(x) = |x|, \;\; \operatorname{co} g = g, \;\; (\operatorname{co} g) \,\square\, \sigma_{\operatorname{dom} f^*}(x) = |x|$

Then

$$\partial_\varepsilon f(x) = \begin{cases} \emptyset & \textit{if } \varepsilon \in [0,1[\ \textit{ and } x \neq 0 \\ [-1,1] & \textit{if } \varepsilon \in [0,1[\ \textit{ and } x = 0 \\ \left[1 - \frac{\varepsilon - 1}{x}, 1\right] & \textit{if } \varepsilon \geq 1 \textit{ and } x > 0 \\ \left[-1, 1 + \frac{1-\varepsilon}{x}\right] & \textit{if } \varepsilon \geq 1 \textit{ and } x < 0 \end{cases},$$

$$\partial_\varepsilon g(x) = \begin{cases} \emptyset & \textit{if } \varepsilon \geq 0 \textit{ and } x \neq 0 \\ \mathbb{R} & \textit{if } \varepsilon \geq 0 \textit{ and } x = 0 \end{cases}.$$

The hypothesis of Theorems 1 and 3 holds but not that of Theorem 2

The convex case

Corollary

Given $f, g : X \to \overline{\mathbb{R}}$. If $f \in \Gamma_0(X)$ and for some $\delta > 0$ we have

$$\partial_\varepsilon f(x) \subset \partial_\varepsilon g(x), \text{ for all } x \in X \text{ and } \varepsilon \in (0, \delta)$$

then

$$f = g + \text{cte}$$

Theorem

Given $f, g : X \to \overline{\mathbb{R}}$. If f si convex and

$$\partial_\varepsilon f(x) \subset \partial_\varepsilon g(x), \text{ for all } x \in X \text{ and } \varepsilon \in (0, \delta)$$

then

$$\operatorname{cl} f = \operatorname{cl} g + \text{cte}$$

Application 1

Let $f : X \to \overline{\mathbb{R}}$ be such that $\operatorname{dom} f^* \neq \emptyset$, and fix $\delta > 0$
We suppose there is $(x_\varepsilon, x_\varepsilon^*)_{0<\varepsilon<\delta} \subset X \times X^*$ such that

$$x_\varepsilon^* \in \partial_\varepsilon f(x_\varepsilon) \quad \text{for each } \varepsilon \in (0, \delta)$$

Then, for any $x \in X$ we have that

$$\overline{\text{co}} f(x) = \sup \left\{ \begin{array}{c} f(x_\varepsilon) + \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle + \langle x_n^*, x - x_n \rangle - \sum_{i=0}^{n} \varepsilon_i \\ \\ \varepsilon \in]0, \delta[, n \in \mathbb{N},\ \varepsilon_i \in]0, \delta[\\ \\ x_i^* \in \partial_{\varepsilon_i} f(x_i), i = 1, \cdots n \end{array} \right\}$$

where $\varepsilon_0 = \varepsilon, x_0 = x_\varepsilon, x_0^* = x_\varepsilon^*$.

Application 2

Let $f, g : X \to \overline{\mathbb{R}}$ be two functions such that $\mathrm{dom}(f+g)^* \neq \emptyset$. Then, the following are equivalent

- $\overline{\mathrm{co}}(f+g) = \overline{\mathrm{co}} f + \overline{\mathrm{co}} g$
- For all $x \in X$ and small $\varepsilon > 0$ it holds

$$\partial_\varepsilon (f+g)(x) = \bigcap_{\delta>0} \mathrm{cl}\left(\bigcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon+\delta \\ \varepsilon_1,\varepsilon_2\geq 0}} \partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} g(x) \right)$$

Thank you!