

# Three integration theorems of the Fenchel subdifferential of nonconvex functions<sup>1</sup>

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<sup>1</sup>Joint work with Y. Garcia and A. Hantoute

# Subdifferential of the inf-convolution

- Our framework is a pair of locally convex spaces paired in duality,  $(X, X^*)$
- Our functions are extended-valued non-necessarily convex
- The (approximate) subdifferential is the classical subdifferential of convex analysis

We start by establishing approximate subdifferential calculus formulas for the inf-convolution, the conjugate function, the closure, the convex envelope, the closed convex envelope

**Proposition :** Let  $f, g : X \rightarrow \overline{\mathbb{R}}$  be two proper functions. Then for every  $x$  and  $\varepsilon \geq 0$  we have that

$$\partial_\varepsilon(f \square g)(x) = \bigcap_{\alpha > 0} \bigcup_{\substack{y \in X \\ \varepsilon_1, \varepsilon_2 > 0, \varepsilon_1 + \varepsilon_2 \leq \varepsilon + \alpha}} \partial_{\varepsilon_1} f(y) \cap \partial_{\varepsilon_2} g(x - y)$$

# Subdifferential of the closure function

Given  $f : X \rightarrow \overline{\mathbb{R}}$  with proper conjugate, for every  $x \in X$  and  $\varepsilon \geq 0$ ,

$$\partial_\varepsilon(\text{cl } f)(x) = \bigcap_{\substack{\delta > 0 \\ V \in \mathcal{V}_x}} \bigcup_{y \in V} \partial_{\varepsilon+\delta} f(y)$$

Equivalently,

$$(\partial_\varepsilon(\text{cl } f))^{-1}(x^*) = \bigcap_{\delta > 0} \overline{(\partial_{\varepsilon+\delta} f)^{-1}(x^*)}$$

Consequently, if  $f : X \rightarrow \overline{\mathbb{R}}$  is convex,

$$\partial_\varepsilon f^*(x^*) = \bigcap_{\delta > 0} \overline{(\partial_{\varepsilon+\delta} f)^{-1}(x^*)}$$

This should be compared with the Fenchel duality in  $\Gamma_0(X)$

$$\partial_\varepsilon f^*(x^*) = (\partial_\varepsilon f)^{-1}(x^*)$$

# Subdifferential of the convex envelope

Given  $f : X \rightarrow \overline{\mathbb{R}}$ , for every  $x \in X$  and  $\varepsilon \geq 0$  we have

$$\partial_\varepsilon(\text{co } f)(x) = \bigcap_{\delta > 0} \bigcup_{\substack{k \in \mathbb{N} \\ (\varepsilon_1, \dots, \varepsilon_k, x_1, \dots, x_k) \in \Delta_k(x, \varepsilon + \delta)}} \bigcap_{i=1, \dots, k} \partial_{\varepsilon_i} f(x_i)$$

where

$$\Delta_k(x, \alpha) := \left\{ (\varepsilon_1, \dots, \varepsilon_k, x_1, \dots, x_k) \mid \varepsilon_i \geq 0, x_i \in X, \exists (\lambda_1, \dots, \lambda_k) \in \Delta_k \right. \\ \left. \text{s.t. } \sum_{i=1, \dots, k} \lambda_i \varepsilon_i \leq \alpha, \sum_{i=1, \dots, k} \lambda_i x_i = x \right\}, \alpha \geq 0$$

Recall that (assuming  $f^*$  proper)

$$(\text{co } f)(x) = \inf \left\{ \sum_{i=1, \dots, k} \lambda_i f(x_i) \mid \sum_{i=1, \dots, k} \lambda_i x_i = x, x_i \in \text{dom } f, \right. \\ \left. (\lambda_1, \dots, \lambda_k) \in \Delta_k, k \in \mathbb{N} \right\}$$

# Subdifferential of the closed convex envelope

For  $f : X \rightarrow \overline{\mathbb{R}}$  with proper conjugate we have, for every  $x \in X$  and  $\varepsilon \geq 0$ ,

$$\partial_\varepsilon(\overline{\text{co}}f)(x) = \bigcap_{\substack{\delta > 0 \\ V \in \mathcal{V}_x}} \bigcup_{\substack{y \in V, k \in \mathbb{N} \\ (\varepsilon_1, \dots, \varepsilon_k, y_1, \dots, y_k) \in \Delta_k(y, \varepsilon + \delta)}} \bigcap_{i=1, \dots, k} \partial_{\varepsilon_i} f(y_i)$$

Consequently, by inverting this formula,

$$\partial_\varepsilon f^*(x^*) = \bigcap_{\delta > 0} \text{cl} \left\{ \sum_{i=1, \dots, k} \lambda_i \partial_{\varepsilon_i} f(y_i) \mid \lambda_i \geq 0, \sum_{i=1, \dots, k} \lambda_i \varepsilon_i \leq \varepsilon + \delta, k \in \mathbb{N} \right\}$$

# Subdifferential of the closed convex envelope - an other formula

Given a function  $f : X \rightarrow \overline{\mathbb{R}}$ , we assume that  $\text{ri}(\text{dom } f^*) \neq \emptyset$ . Then for every  $x \in X$  we have that

$$\partial(\overline{\text{co}}f)(x) = \bigcap_{\varepsilon > 0, V \in \mathcal{V}_x} \bigcup_{\substack{(\lambda_1, \dots, \lambda_k) \in \Delta_k, k \in \mathbb{N} \\ \sum_{i=1, \dots, k} \lambda_i y_i + z \in V, z \in X}} \bigcap_{i=1, k} \partial_\varepsilon f(y_i) \cap \partial \sigma_{\text{dom } f^*}(z)$$

Equivalently, we have that

$$\partial(\overline{\text{co}}f)(x) = \bigcap_{\varepsilon > 0, V \in \mathcal{V}_x} \bigcup_{\substack{(\lambda_1, \dots, \lambda_k) \in \Delta_k, k \in \mathbb{N} \\ \sum_{i=1, \dots, k} \lambda_i y_i + z \in V, z \in X}} \bigcap_{i=1, k} \partial_\varepsilon f(y_i) \cap \partial f^\infty(z)$$

where  $f^\infty(x) := \liminf_{s \rightarrow 0^+, \langle y, \cdot \rangle \rightarrow \langle x, \cdot \rangle \text{ on } \text{dom } f^*} sf(s^{-1}y)$

# Integration theorems

Given  $f, g : X \rightarrow \overline{\mathbb{R}}$  with  $f^*$  being proper. For  $\delta \in (0, \infty]$  we suppose

$$\partial_\varepsilon f(x) \subset \partial_\varepsilon g(x), \quad \text{for all } x \in X \text{ and all } \varepsilon \in (0, \delta)$$

**Theorem 1:** If  $\delta$  is finite, then

$$f^* = g^* + I_{\text{dom } f^*} + \text{cte}$$

**Theorem 2:** If  $\delta = \infty$ , then

$$\overline{\text{co}}f = \overline{\text{co}}g + \text{cte}$$

**Theorem 3:** If  $\delta$  is finite, then

$$\overline{\text{co}}f = \sup_{L \in \mathcal{F}} \overline{(\overline{\text{co}}g) \square \sigma_{\text{dom } f^* \cap L}} + \text{cte}$$

where  $\mathcal{F}$  refers to the family of finite-dimensional subspaces of  $X^*$ .

In particular, if  $\text{ri}(\text{dom } f^*) \neq \emptyset$ , or if  $g^*$  is continuous at some point in  $\text{dom } f^*$ , or if  $\text{dom } f^*$  is closed, then

$$\overline{\text{co}}f = \overline{(\overline{\text{co}}g) \square \sigma_{\text{dom } f^*}} + \text{cte}$$

# Example

Consider the functions  $f, g : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$  defined as

$$g(x) := \mathbb{I}_{\{0\}} \text{ and } f(x) := \begin{cases} |x| + 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

- $f^* = \mathbb{I}_{[-1,1]} - 1$ ,  $g^* \equiv 0$
- $(\text{co } f)(x) = |x|$ ,  $\text{co } g = g$ ,  $(\text{co } g) \square \sigma_{\text{dom } f^*}(x) = |x|$

Then

$$\partial_\varepsilon f(x) = \begin{cases} \emptyset & \text{if } \varepsilon \in [0, 1[ \text{ and } x \neq 0 \\ [-1, 1] & \text{if } \varepsilon \in [0, 1[ \text{ and } x = 0 \\ \left[1 - \frac{\varepsilon-1}{x}, 1\right] & \text{if } \varepsilon \geq 1 \text{ and } x > 0 \\ \left[-1, 1 + \frac{1-\varepsilon}{x}\right] & \text{if } \varepsilon \geq 1 \text{ and } x < 0 \end{cases},$$
$$\partial_\varepsilon g(x) = \begin{cases} \emptyset & \text{if } \varepsilon \geq 0 \text{ and } x \neq 0 \\ \mathbb{R} & \text{if } \varepsilon \geq 0 \text{ and } x = 0 \end{cases}.$$

**The hypothesis of Theorems 1 and 3 holds but not that of Theorem 2**

# The convex case

## Corollary

Given  $f, g : X \rightarrow \overline{\mathbb{R}}$ . If  $f \in \Gamma_0(X)$  and for some  $\delta > 0$  we have

$$\partial_\varepsilon f(x) \subset \partial_\varepsilon g(x), \text{ for all } x \in X \text{ and } \varepsilon \in (0, \delta)$$

then

$$f = g + \text{cte}$$

## Theorem

Given  $f, g : X \rightarrow \overline{\mathbb{R}}$ . If  $f$  is convex and

$$\partial_\varepsilon f(x) \subset \partial_\varepsilon g(x), \text{ for all } x \in X \text{ and } \varepsilon \in (0, \delta)$$

then

$$\text{cl } f = \text{cl } g + \text{cte}$$

# Application 1

Let  $f : X \rightarrow \overline{\mathbb{R}}$  be such that  $\text{dom } f^* \neq \emptyset$ , and fix  $\delta > 0$

We suppose there is  $(x_\varepsilon, x_\varepsilon^*)_{0 < \varepsilon < \delta} \subset X \times X^*$  such that

$$x_\varepsilon^* \in \partial_\varepsilon f(x_\varepsilon) \quad \text{for each } \varepsilon \in (0, \delta)$$

Then, for any  $x \in X$  we have that

$$\overline{\text{co}}f(x) = \sup \left\{ \begin{array}{l} f(x_\varepsilon) + \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle + \langle x_n^*, x - x_n \rangle - \sum_{i=0}^n \varepsilon_i \\ \varepsilon \in ]0, \delta[, n \in \mathbb{N}, \varepsilon_i \in ]0, \delta[ \\ x_i^* \in \partial_{\varepsilon_i} f(x_i), i = 1, \dots, n \end{array} \right\}$$

where  $\varepsilon_0 = \varepsilon, x_0 = x_\varepsilon, x_0^* = x_\varepsilon^*$ .

## Application 2

Let  $f, g : X \rightarrow \overline{\mathbb{R}}$  be two functions such that  $\text{dom}(f + g)^* \neq \emptyset$ . Then, the following are equivalent

- $\overline{\text{co}}(f + g) = \overline{\text{co}}f + \overline{\text{co}}g$
- For all  $x \in X$  and small  $\varepsilon > 0$  it holds

$$\partial_\varepsilon(f + g)(x) = \bigcap_{\delta > 0} \text{cl} \left( \bigcup_{\substack{\varepsilon_1 + \varepsilon_2 = \varepsilon + \delta \\ \varepsilon_1, \varepsilon_2 \geq 0}} \partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} g(x) \right)$$

Thank you!