Terryfest 2015, Limoges

ω-limit sets for differential inclusions

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Based on joint work with M. Krastanov and V. Veliov

Supported by NSF Grant DMS-1008341

Motivation: nonsmooth/discontinuous feedback

Arsie, A., Ebenbauer, C., Locating omega-limit sets using height functions. J. Differential Equations 248, 2458–2469 (2010)

$f : \boldsymbol{R}^n \to \boldsymbol{R}^n$ locally Lipschitz continuous.

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \tag{1}$$

Carathéodory solutions on $[0, +\infty)$: a function $\varphi : [0, +\infty) \to \boldsymbol{R}^n$ which is absolutely continuous and satisfies (1) for a.e. $t \in [0, +\infty)$.

ω-limit set $\omega(x_0)$: the collection of points $y \in \boldsymbol{R}^n$ for each of which there exists a Carathéodory solution $\varphi(\cdot, x_0)$ of (1) which is bounded on $[0, +\infty)$, and a sequence $t_k \to \infty$ such that $\varphi(t_k, x_0) \to y$ as $k \to \infty$.

Theorem (Arsie, A., Ebenbauer (2010).

Assume we are given a closed set $\mathcal{S} \subset \boldsymbol{R}^n$ which contains $\omega(x_0)$ and a function $V : G \to \boldsymbol{R}$ which is continuously differentiable over a neighborhood of $\mathcal{S}$. Define $\mathcal{U} := \{x \in \mathcal{S} : \dot{V}_f(x) < 0\}$ and assume that $V(\mathcal{S} \setminus \mathcal{U})$ does not contain any open interval. Then the ω-limit set $\omega(x_0)$ is contained in a connected subset of the set $\mathcal{S} \setminus \mathcal{U}$.

Differential inclusion

(2) $$\dot{x}(t) \in F(t, x(t)), \quad x(0) = x_0$$

STANDING ASSUMPTION. For every $x_0 \in \boldsymbol{R}^n$ there exist positive reals r and M such that

$$\|F(t,x)\| \leq M \quad \text{for every } x \in \boldsymbol{B}_r(x_0) \text{ and every } t \geq 0.$$

ω-limit set $\omega(x_0)$: nonempty if, e.g., F is either upper semi-continuous with compact convex values or lower semi-continuous, and an appropriate growth condition holds.

The upper Dini directional derivative of a function $V : \boldsymbol{R}^n \to \boldsymbol{R}$ at x in the direction l is

$$D^+V(x;l) := \limsup_{h \searrow 0} \frac{V(x+hl) - V(x)}{h}.$$

Localization of the ω-limit set

Theorem.

Let $\mathcal{S}$ be a closed subset of $\boldsymbol{R}^n$, $\mathcal{U}$ be a relatively open subset of $\mathcal{S}$, G be an open set containing $\mathcal{S}$ and let $Z := (G \setminus \mathcal{S}) \cup \mathcal{U}$.
Let $V : G \to \boldsymbol{R}$ be locally Lipschitz and $W : Z \to \boldsymbol{R}$ be lower semicontinuous and suppose that the following conditions hold:

(B1) For every $\varepsilon > 0$ and for each bounded solution $\varphi(\cdot, x_0)$ of (2) there exists $T > 0$ such that $\mathrm{dist}(\varphi(t, x_0), \mathcal{S}) < \varepsilon$ for every $t \geq T$;

(B2) $W(x) > 0$ for every $x \in \mathcal{U}$;

(B3) $\sup_{v \in F(t,x)} D^+V(x; v) \leq -W(x)$ for every $x \in Z$;

(B4) Every open interval contained in $V(\mathcal{S} \setminus \mathcal{U})$ has empty intersection with $V(\mathcal{U})$.

Then the set $\omega(x_0)$ is contained in $\mathcal{S} \setminus \mathcal{U}$.

Sketch of proof

On the contrary, assume there exists $\bar{x} \in \omega(x_0) \cap \mathcal{U}$. Then prove that there exists

$$c \in V(\bar{x} - \varepsilon, V(\bar{x}) + \varepsilon)$$

for a specially chosen ε (sufficiently small) such that

$$\{x \in \mathcal{S} + \delta B) \cap K \mid V(x) = c\} \subset Z \cup \{x \mid W(x) > 0\}.$$

Take a sequence $t_k \to \infty$ and estimate $V(\varphi(t)$ from above by $V(\bar{x})$. Then show that

$$V(\varphi(t) < c \quad \text{for} \quad t \geq t_k + \tau$$

for a specially chosen τ.
Obtain contradiction by using the assumption for W.

THANK YOU!