Optimal pits and optimal transportation

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Open Pit Mining

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Mining Processes

1. **Open-pit mine**
2. **Crushing and grinding**
3. **Crushing**
4. **Leaching**
5. **Solvent extraction**
6. **Electrowinning**
7. **Gases to cleaning plant**
8. **Gas cleaning plant**
9. **Sale of acid**
10. **Sale of white metal, waste rock and blister**

**Processes**
- **Underground mine**
- **Purchase of copper concentrates**
- **Sale of molybdenum**
- **Sale of copper concentrates**
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- **Purchase of anodes**
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- **Electrowinning**
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Open Pit Mine Planning

1. Project evaluation: is it worth investing?
   ▶ Where to dig? How deep? What to process?
   Optimum open pit problem (determining ultimate pit limits)

2. Rough-cut planning: take time into account
   ▶ Where, when and what to excavate, to process subject to capacity and other resource constraints, and the time value of money (cash flows)
   ▶ Process choices, major equipment decisions
   Mine production planning problem (decisions over time)

3. Detailed operations planning
   ▶ Detailed mine design: benches, routes, facilities
   ▶ Operations scheduling, flows of materials, etc.

4. Execution...
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(25 million tons rock slide, 2006)
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Bingham Canyon copper mine, Utah
(massive landslide, 10 April 2013)
Discretization: Block Models

[Lerchs and Grossmann, 1965]
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Divide the volume of interest into 3D blocks
  ▶ typically rectangular, with vertical sides

Leads to a nicely structured (dual network flow, minimum cut) discrete optimization problem

▶ implemented in commercial software (Whittle, Geovia)
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Discretized vs Continuous Models?

Discretized (block) models:
▶ are very large (100,000s to millions of blocks)
▶ production planning models even larger (× number of periods)
▶ the real problem is, to a large extent, continuous:
▶ ore density and rock properties tend to vary continuously
▶ their distributions are estimated ("smoothed") from sample (drill hole) data and other geological information
▶ block precedences only roughly model the slope constraints

Earlier continuous space models:
▶ Matheron (1975) (focus on "cutoff grade" parametrization)
▶ determine optimum depth \( \varphi(y) \) under each surface point \( y \)
s.t. bounds on the derivative of \( \varphi \) (wall slope constraints)

All these continuous space approaches suffer from lack of convexity
▶ how to deal with local optima?
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Perspectives
A general model

\[ E \subset \mathbb{R}^3 \]: the domain to be mined
e.g., \[ E = A \times [h_1, h_2] \], where \( A \subset \mathbb{R}^2 \) is the claim

\[ h_1, h_2 \] is the elevation or depth range

\[ \Gamma \]: extracting \( x \) requires extracting all of \( \Gamma(x) \)

\[ \Gamma \] is transitive:
\[ x' \in \Gamma(x) \] and \( x'' \in \Gamma(x') \) \( \Rightarrow \)
\[ x'' \in \Gamma(x) \]

\[ \Gamma \] is reflexive:
\[ x \in \Gamma(x) \]

\[ \Gamma \] is closed graph:
\[ \{ (x, y) : x \in E, y \in \Gamma(x) \} \] is closed

\( F \) is a measurable subset of \( E \) closed under \( \Gamma \):
\[ \Gamma(F) = F \]

\[ \Gamma(F) := \bigcup_{x \in F} \Gamma(x) \]

\( g \): a continuous function

\[ g(x) \]: net profit from volume element \( dx \)
\[ g(F) := \int_F g(x) \, dx \]

\[ \int_E \max\{0, g(x)\} \, dx > 0 \] (there is some profit to be made)

Optimum pit problem:
find \( F^* \in \arg \max \{ g(F) : F \text{ is a pit} \} \)
A general model [Matheron 1975]: Given

- compact $E \subset \mathbb{R}^3$: the domain to be mined
  - e.g., $E = A \times [h_1, h_2]$, where $A \subset \mathbb{R}^2$ is the *claim*
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Open Pit Problem: a Continuous Space Model

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- map $\Gamma : E \rightarrow E$: extracting $x$ requires extracting all of $\Gamma(x)$
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- continuous function \( g : E \rightarrow \mathbb{R} \)

\( g(x) \, dx \) net profit from volume element \( dx \)

\( g(F) := \int_F g(x) \, dx \) total net profit from pit \( F \)

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$$\Gamma(F) = F \quad \text{where} \quad \Gamma(F) := \bigcup_{x \in F} \Gamma(x)$$

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Open Pit Problem: a Continuous Space Model

**A general model** [Matheron 1975]: Given

- compact $E \subset \mathbb{R}^3$: the domain to be mined
  - e.g., $E = A \times [h_1, h_2]$, where $A \subset \mathbb{R}^2$ is the *claim*
    - $[h_1, h_2]$ is the elevation or depth range

- map $\Gamma : E \rightarrow E$: extracting $x$ requires extracting all of $\Gamma(x)$
  - transitive: $[x' \in \Gamma(x) \text{ and } x'' \in \Gamma(x')] \implies x'' \in \Gamma(x)$
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Optimum pit problem: find $F^* \in \arg \max\{g(F) : F \text{ is a pit}\}$
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- A Continuous Space Model
- An Optimal Transportation Problem
- The Kantorovich Dual
- Elements of \( c \)-Convex Analysis
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Add a sink $\omega$ unallocated profits from excavated points will be sent to $\omega$ and a source $\alpha$ unallocated costs of unexcavated points will be paid by $\alpha$.

Let $X := E^+ \cup \{ \alpha \}$ and $Y := E^- \cup \{ \omega \}$ (also compact).

 endowed with non-negative measures $\mu$ and $\nu$ defined by

$$
\mu(\{ \alpha \}) = \int_{E^-} |g(z)| \mu(\{ \alpha \}) = \int_{E^+} g(z) \mu(\{ \alpha \})
$$

$$
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$$

Profit allocations are allowed from every profitable $x \in E^+$ to every $y \in \Gamma(x) \cap E^-$ from source $\alpha$ to all $y \in E^-$ (unpaid costs) from all $x \in E^+$ to sink $\omega$ (unallocated, or "excess" profits).

These restrictions will be modelled by a "transportation" (or allocation) cost function $c : X \times Y \to \mathbb{R}$.
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Set $\Pi(\mu, \nu)$ of nonnegative Radon measures (profit allocations) $\pi$ with marginals $\pi_X = \mu$ and $\pi_Y = \nu$
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**Optimal transportation problem** in Kantorovich form:
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$$
\min_{\pi} \mathbb{E}^\pi[c] := \int_{X \times Y} c(x, y) d\pi \quad \text{s.t. } \pi \in \Pi(\mu, \nu) \quad (K)
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**Proposition 1:** Problem \((K)\) has a solution
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**Proposition 1:** Problem $(K)$ has a solution

**Proof:** The set of positive Radon measures on compact space $X \times Y$ is weak-* compact, and the map $\pi \rightarrow E^\pi[c]$ is weak-* l.s.c. □
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Potentials (duals, Lagrange multipliers)
- $p \in L^1(X, \mu)$ associated with $\pi_X = \mu$
- $q \in L^1(Y, \nu)$ associated with $\pi_Y = \nu$
The Kantorovich Dual

Potentials (duals, Lagrange multipliers)

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Dual admissible set:
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Dual objective:
\[
J(p, q) := \int_X p \, d\mu - \int_Y q \, d\nu
\]
\[
= \int_{E^+} (p(\omega) - q(\omega)) \, d\mu - \int_{E^-} (q(\omega) - p(\alpha)) \, d\nu
\]

Theorem [Kantorovich, 1942]: When the cost function $c$ is l.s.c.,
\[
\inf(D) = \sup(D)
\]
there is no duality gap (in continuous variables).
The Kantorovich Dual

**Potentials** (duals, Lagrange multipliers)

- \( p \in L^1(X, \mu) \) associated with \( \pi_X = \mu \)
- \( q \in L^1(Y, \nu) \) associated with \( \pi_Y = \nu \)

Dual admissible set:
\[
A := \{ (p, q) : p(x) - q(y) \leq c(x, y) \ (\mu, \nu) \text{-a.s.} \}
\]

Dual objective:
\[
J(p, q) := \int_X p \, d\mu - \int_Y q \, d\nu
\]

\[
= \int_{E^+} (p(z) - q(\omega)) \, d\mu - \int_{E^-} (q(z) - p(\alpha)) \, d\nu
\]

Kantorovich dual:
\[
\sup J(p, q) \text{ s.t. } (p, q) \in A \quad (D)
\]
The Kantorovich Dual

Potentials (duals, Lagrange multipliers)
- $p \in L^1(X, \mu)$ associated with $\pi_X = \mu$
- $q \in L^1(Y, \nu)$ associated with $\pi_Y = \nu$

Dual admissible set:
$$A := \{(p, q) : p(x) - q(y) \leq c(x, y) \ (\mu, \nu)\text{-a.s.}\}$$

Dual objective:
$$J(p, q) := \int_X p \, d\mu - \int_Y q \, d\nu$$
$$= \int_{E^+} (p(z) - q(\omega)) \, d\mu - \int_{E^-} (q(z) - p(\alpha)) \, d\nu$$

Kantorovich dual: $\sup J(p, q) \text{ s.t. } (p, q) \in A$(D)

Theorem [Kantorovich, 1942]: *When the cost function $c$ is l.s.c.,* $\inf(K) = \sup(D)$
The Kantorovich Dual

**Potentials** (duals, Lagrange multipliers)

- \( p \in L^1(X, \mu) \) associated with \( \pi_X = \mu \)
- \( q \in L^1(Y, \nu) \) associated with \( \pi_Y = \nu \)

Dual admissible set:

\[
\mathcal{A} := \{(p, q) : p(x) - q(y) \leq c(x, y) \ (\mu, \nu)\text{-a.s.}\}
\]

Dual objective:

\[
J(p, q) := \int_X p \, d\mu - \int_Y q \, d\nu
\]

\[
= \int_{E^+} (p(z) - q(\omega)) \, d\mu - \int_{E^-} (q(z) - p(\alpha)) \, d\nu
\]

Kantorovich dual:

\[
\sup J(p, q) \quad \text{s.t.} \quad (p, q) \in \mathcal{A} \\
(D)
\]

**Theorem** [Kantorovich, 1942]: When the cost function \( c \) is l.s.c.,

\[
\inf(K) = \sup(D)
\]

- there is no duality gap (in continuous variables)
Connection to the Optimum Pit Problem

Let \( F \) be a pit, \( F^+ := F \cap E^+ \) and \( F^- := F \cap E^- \).

Define \( p_F : X \rightarrow \mathbb{R} \) and \( q_F : Y \rightarrow \mathbb{R} \) by:

\[
p_F(\alpha) = 0, \quad p_F(x) = \begin{cases} 
1 & \text{if } x \in F^+ \\
0 & \text{otherwise}
\end{cases}
\]

\[
q_F(\omega) = 0, \quad q_F(y) = \begin{cases} 
1 & \text{if } y \in F^- \\
0 & \text{otherwise}
\end{cases}
\]

Then \((p_F, q_F)\) is admissible (i.e., in \( A \)) and \( J(p_F, q_F) = g(F) \).

Corollary:

\[\sup(P) \leq \inf(K)\]

i.e., transportation problem \((K)\) is a weak dual to the optimum pit problem \((P)\).
Let $F$ be a pit, $F^+ := F \cap E^+$ and $F^- := F \cap E^-$
Let $F$ be a pit, $F^+ := F \cap E^+$ and $F^- := F \cap E^-$

Define $p_F : X \to \mathbb{R}$ and $q_F : Y \to \mathbb{R}$ by:

$p_F(\alpha) = 0$, $p_F(x) = \begin{cases} 1 & \text{if } x \in F^+ \\ 0 & \text{otherwise} \end{cases}$

$q_F(\omega) = 0$, $q_F(y) = \begin{cases} 1 & \text{if } y \in F^- \\ 0 & \text{otherwise} \end{cases}$
Let $F$ be a pit, $F^+ := F \cap E^+$ and $F^- := F \cap E^-$.

Define $p_F : X \to \mathbb{R}$ and $q_F : Y \to \mathbb{R}$ by:

$$p_F(\alpha) = 0, \quad p_F(x) = \begin{cases} 1 & \text{if } x \in F^+ \\ 0 & \text{otherwise} \end{cases}$$

$$q_F(\omega) = 0, \quad q_F(y) = \begin{cases} 1 & \text{if } y \in F^- \\ 0 & \text{otherwise} \end{cases}$$

Then $(p_F, q_F)$ is admissible (i.e., in $\mathcal{A}$) and $J(p_F, q_F) = g(F)$.
Let $F$ be a pit, $F^+ := F \cap E^+$ and $F^- := F \cap E^-$

Define $p_F : X \to \mathbb{R}$ and $q_F : Y \to \mathbb{R}$ by:

$$p_F(\alpha) = 0, \quad p_F(x) = \begin{cases} 1 & \text{if } x \in F^+ \\ 0 & \text{otherwise} \end{cases}$$

$$q_F(\omega) = 0, \quad q_F(y) = \begin{cases} 1 & \text{if } y \in F^- \\ 0 & \text{otherwise} \end{cases}$$

Then $(p_F, q_F)$ is admissible (i.e., in $\mathcal{A}$) and $J(p_F, q_F) = g(F)$

**Corollary:** $\sup(P) \leq \inf(K)$
Let $F$ be a pit, $F^+ := F \cap E^+$ and $F^- := F \cap E^-$

Define $p_F : X \to \mathbb{R}$ and $q_F : Y \to \mathbb{R}$ by:

$$p_F(\alpha) = 0, \quad p_F(x) = \begin{cases} 1 \text{ if } x \in F^+ \\ 0 \text{ otherwise} \end{cases}$$

$$q_F(\omega) = 0, \quad q_F(y) = \begin{cases} 1 \text{ if } y \in F^- \\ 0 \text{ otherwise} \end{cases}$$

Then $(p_F, q_F)$ is admissible (i.e., in $A$) and $J(p_F, q_F) = g(F)$

**Corollary:** $\sup(P) \leq \inf(K)$

- i.e., transportation problem (K) is a *weak dual* to the optimum pit problem (P)
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Given $c: X \times Y \to \mathbb{R}$, define the $c$-Fenchel conjugates (or $c$-Fenchel-Legendre transforms) of any function $p \in L^1(X, \mu)$ by

$$p^\#(y) := \text{ess sup}_{x \in X} (p(x) - c(x,y))$$

and of any function $q \in L^1(Y, \nu)$ by

$$q^\flat(x) := \text{ess inf}_{y \in Y} (q(y) + c(x,y))$$

where $\text{ess sup} f(x) = \inf_{N \in \mathbb{N}} \sup_{x \in X \setminus N} f(x)$, where $N$ is the set of measurable subsets $N \subset X$ with $\mu(N) = 0$.

To simplify, we'll write $\sup$ and $\inf$ instead of $\text{ess sup}$ and $\text{ess inf}$.

Similarly, all equalities and inequalities will be $\mu$-a.e. in $X$ and $\nu$-a.e. in $Y$. 

---

$C$-Fenchel Conjugates
Given $c : X \times Y \to \mathbb{R}$, define the $c$-Fenchel conjugates (or $c$-Fenchel-Legendre transforms)

- $p^\#: Y \to \mathbb{R}$ of any function $p \in L^1(X, \mu)$ by
  $$p^\#(y) := \text{ess sup}_{x \in X} (p(x) - c(x, y))$$

- $q^\flat : X \to \mathbb{R}$ of any function $q \in L^1(Y, \nu)$ by
  $$q^\flat(x) := \text{ess inf}_{y \in Y} (q(y) + c(x, y))$$

where $\text{ess sup} f(x) = \inf_{N \in \mathcal{N}} \sup_{x \in X \setminus N} f(x)$, where $\mathcal{N}$ is the set of measurable subsets $N \subset X$ with $\mu(N) = 0$
Given \( c : X \times Y \to \mathbb{R} \), define the \( c \)-Fenchel conjugates (or \( c \)-Fenchel-Legendre transforms)

- \( p^\# : Y \to \mathbb{R} \) of any function \( p \in L^1(X, \mu) \) by
  \[
  p^\#(y) := \text{ess sup}_{x \in X} (p(x) - c(x, y))
  \]

- \( q^\flat : X \to \mathbb{R} \) of any function \( q \in L^1(Y, \nu) \) by
  \[
  q^\flat(x) := \text{ess inf}_{y \in Y} (q(y) + c(x, y))
  \]

where \( \text{ess sup} f(x) = \inf_{N \in \mathcal{N}} \sup_{x \in X \setminus N} f(x) \), where \( \mathcal{N} \) is the set of measurable subsets \( N \subset X \) with \( \mu(N) = 0 \)

- To simplify, we’ll write \( \sup \) and \( \inf \) instead of \( \text{ess sup} \) and \( \text{ess inf} \)
- Similarly, all equalities and inequalities will be \( \mu \)-a.e. in \( X \) and \( \nu \)-a.e. in \( Y \)
Properties of $c$-Fenchel Conjugates

[Carlier, 2003; Ekeland, 2010]
Properties of $c$-Fenchel Conjugates

[Carlier, 2003; Ekeland, 2010]

For all $x \in X$, $y \in Y$,

$$p(x) \leq c(x, y) + p^\#(y) \leq p^{\#\#}(x)$$

$$q(y) \geq q^\flat(x) - c(x, y) \geq q^{\flat\#}(y)$$
Properties of $c$-Fenchel Conjugates

[Carlier, 2003; Ekeland, 2010]

For all $x \in X$, $y \in Y$,

\[ p(x) \leq c(x, y) + p^\#(y) \leq p^{\#\#}(x) \]
\[ q(y) \geq q^b(x) - c(x, y) \geq q^{\#\#}(y) \]

c-Fenchel duality:

\[ p^{\#\#} = p^\# \quad \text{and} \quad q^{\#\#} = q^b \]
Properties of $c$-Fenchel Conjugates

[Carlier, 2003; Ekeland, 2010]

For all $x \in X$, $y \in Y$, 

$$p(x) \leq c(x, y) + p^\#(y) \leq p^{\#b}(x)$$

$$q(y) \geq q^b(x) - c(x, y) \geq q^{\#b}(y)$$

$c$-Fenchel duality:

$$p^{\#b\#} = p^\# \quad \text{and} \quad q^{b\#b} = q^b$$

Monotonicity:

$$p_1 \leq p_2 \implies p_1^\# \leq p_2^\#$$

$$q_1 \leq q_2 \implies q_1^b \leq q_2^b$$
$c$-Fenchel Transforms for the Open Pit Dual Problem

\[ p^\#(y) := \max \{ p(\alpha), \sup_{x \in \Gamma(y)} p(x) \} \quad \text{for} \quad y \in E \]

\[ q^\flat(x) := \min \{ 1 + q(\omega), \inf_{y \in \Gamma(x)} q(y) \} \quad \text{for} \quad x \in E^+ \]

$p^\#$ and $q^\flat$ are increasing with respect to $\Gamma$:

\[ x' \in \Gamma(x) \Rightarrow q^\flat(x') \geq q^\flat(x) \]

\[ y' \in \Gamma(y) \Rightarrow p^\#(y') \geq p^\#(y) \]
$p^\#(y) := \max \left\{ p(\alpha), \sup_{x : y \in \Gamma(x)} p(x) \right\}$ for $y \in E^-$

$p^\#(\omega) := \max \left\{ p(\alpha), \sup_{x \in E^+} p(x) - 1 \right\}$

$q^\flat(x) := \min \left\{ 1 + q(\omega), \inf_{y \in \Gamma(x)} q(y) \right\}$ for $x \in E^+$

$q^\flat(\alpha) := \min \left\{ q(\omega), \inf_{y \in E^-} q(y) \right\}$
$p^\#(y) := \max \left\{ p(\alpha), \sup_{x : y \in \Gamma(x)} p(x) \right\}$ for $y \in E^-$

$p^\#(\omega) := \max \left\{ p(\alpha), \sup_{x \in E^+} p(x) - 1 \right\}$

$q^\flat(x) := \min \left\{ 1 + q(\omega), \inf_{y \in \Gamma(x)} q(y) \right\}$ for $x \in E^+$

$q^\flat(\alpha) := \min \left\{ q(\omega), \inf_{y \in E^-} q(y) \right\}$

$p^\#$ and $q^\flat$ are increasing with respect to $\Gamma$:

$x' \in \Gamma(x) \implies q^\flat(x') \geq q^\flat(x)$

$y' \in \Gamma(y) \implies p^\#(y') \geq p^\#(y)$
\( p^\#(y) := \max \left\{ p(\alpha), \sup_{x : y \in \Gamma(x)} p(x) \right\} \) for \( y \in E^- \)

\( p^\#(\omega) := \max \left\{ p(\alpha), \sup_{x \in E^+} p(x) - 1 \right\} \)

\( q^\flat(x) := \min \left\{ 1 + q(\omega), \inf_{y \in \Gamma(x)} q(y) \right\} \) for \( x \in E^+ \)

\( q^\flat(\omega) := \min \left\{ q(\omega), \inf_{y \in E^-} q(y) \right\} \)

\( p^\# \) and \( q^\flat \) are increasing with respect to \( \Gamma \):

\( x' \in \Gamma(x) \implies q^\flat (x') \geq q^\flat (x) \)

\( y' \in \Gamma(y) \implies p^\# (y') \geq p^\# (y) \)

For a pit \( F \), \( p_F = q^\flat_F \) and \( q_F = p^\#_F \).
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Translation Invariance

Given \((p, q) \in A\) and constants \(p_0, p_1, q_0, q_1\) satisfying:

\[
\mu(E + q_0 - p_1) - \nu(E - p_0 + q_1) = 0
\]

define \(\tilde{p}\) and \(\tilde{q}\) by:

\[
\tilde{p}(\alpha) = p(\alpha) - p_0 \\
\tilde{p}(x) = p(x) - p_1 \quad \text{for} \quad x \in E^+ \\
\tilde{q}(\omega) = q(\omega) - q_0 \\
\tilde{q}(y) = q(y) - q_1 \quad \text{for} \quad y \in E^-
\]

Then:

\[
J(\tilde{p}, \tilde{q}) = J(p, q)
\]
Translation Invariance

Given \((p, q) \in \mathcal{A}\) and constants \(p_0, p_1, q_0, q_1\) satisfying:

\[
\mu \left( E^+ \right) (q_0 - p_1) - \nu \left( E^- \right) (p_0 - q_1) = 0
\]

define \(\tilde{p}\) and \(\tilde{q}\) by:

\[
\begin{align*}
\tilde{p}(\alpha) &= p(\alpha) - p_0 \\
\tilde{p}(x) &= p(x) - p_1 \quad \text{for } x \in E^+ \\
\tilde{q}(\omega) &= q(\omega) - q_0 \\
\tilde{q}(y) &= q(y) - q_1 \quad \text{for } y \in E^-
\end{align*}
\]

Then:

\[
J(\tilde{p}, \tilde{q}) = J(p, q)
\]
\( c \)-Fenchel Transforms Give Local Improvements

If \((p,q) \in A\), then

\[ p(x) - q(y) \leq c(x,y) \]

for all \((x,y)\), so that:

\[ p(x) \leq \inf_y \{ c(x,y) + q(y) \} = q^\flat(x) \]

\[ q(y) \geq \sup_x \{ p(x) - c(x,y) \} = p^\sharp(y) \]

Therefore \((p,p^\sharp) \in A\) and

\[ J(p,p^\sharp) \geq J(p,q)\]

\((q^\flat,q^\flat) \in A\) and

\[ J(q^\flat,q^\flat) \geq J(p,q)\]

This implies

\[ J(p,q) \leq J(p,p^\sharp) \leq J(p^\flat^\sharp,p^\sharp) \]

Letting \( \bar{p} := p^\flat^\sharp \) and \( \bar{q} := p^\sharp \), we get:

\[ J(p,q) \leq J(\bar{p},\bar{q}) \]

\( \bar{p} = q^\flat \) and \( \bar{q} = p^\sharp \).
If \((p, q) \in A\), then \(p(x) - q(y) \leq c(x, y)\) for all \((x, y)\), so that:

\[
p(x) \leq \inf_y \{c(x, y) + q(y)\} = q^b(x)
\]

\[
q(y) \geq \sup_x \{p(x) - c(x, y)\} = p^\#(y)
\]
If \((p, q) \in A\), then \(p(x) - q(y) \leq c(x, y)\) for all \((x, y)\), so that:

\[
p(x) \leq \inf_y \{c(x, y) + q(y)\} = q^b(x)
\]

\[
q(y) \geq \sup_x \{p(x) - c(x, y)\} = p^\#(y)
\]

Therefore

\[
(p, p^\#) \in A \quad \text{and} \quad J(p, p^\#) \geq J(p, q)
\]

\[
(q^b, q) \in A \quad \text{and} \quad J(q^b, q) \geq J(p, q)
\]
If \((p, q) \in \mathcal{A}\), then \(p(x) - q(y) \leq c(x, y)\) for all \((x, y)\), so that:

\[
p(x) \leq \inf_y \{c(x, y) + q(y)\} = q^b(x)
\]

\[
q(y) \geq \sup_x \{p(x) - c(x, y)\} = p^\#(y)
\]

Therefore

\[
(p, p^\#) \in \mathcal{A} \quad \text{and} \quad J(p, p^\#) \geq J(p, q)
\]

\[
(q^b, q) \in \mathcal{A} \quad \text{and} \quad J(q^b, q) \geq J(p, q)
\]

This implies

\[
J(p, q) \leq J(p, p^\#) \leq J(p^b, p^\#)
\]
If \((p, q) \in A\), then \(p(x) - q(y) \leq c(x, y)\) for all \((x, y)\), so that:

\[
p(x) \leq \inf_y \{c(x, y) + q(y)\} = q^b(x)
\]

\[
q(y) \geq \sup_x \{p(x) - c(x, y)\} = p^\#(y)
\]

Therefore

\[
(p, p^\#) \in A \quad \text{and} \quad J(p, p^\#) \geq J(p, q)
\]

\[
(q^b, q) \in A \quad \text{and} \quad J(q^b, q) \geq J(p, q)
\]

This implies

\[
J(p, q) \leq J(p, p^\#) \leq J(p^{\#b}, p^\#)
\]

Letting \(\bar{p} := p^{\#b}\) and \(\bar{q} := p^\#\), we get:

\[
J(p, q) \leq J(\bar{p}, \bar{q})
\]

\[
\bar{p} = q^b \quad \text{and} \quad \bar{q} = p^\#
\]
A Dual Solution

Proposition 2: Problem (D) has a solution $(\bar{p}, \bar{q})$ with $\bar{p} = \bar{q} \leq 0 < \overline{p} \leq 1$, $\bar{q} = \bar{p} \leq 0 < \overline{q} \leq 1$, and $\inf y \in E - q_n(y) = 0$.

Proof: Take a maximizing sequence $(p_n, q_n) \in A$. By preceding results, we may assume $p_n = q_n^{\flat}$ and $q_n = p_n^{\sharp} = 0$, and $\inf y \in E - q_n(y) = 0$. Then, for all $x \in E^+$, $p_n(x) = \min\{1, \inf y \in \Gamma(x) \cap E - q_n(y)\}$. This implies $0 \leq p_n(x) \leq 1$. Similarly, we get $0 \leq q_n(x) \leq 1$. So the family $(p_n, q_n)$ is equi-integrable in $L^1(\mu) \times L^1(\nu)$.

By the Dunford-Pettis Theorem, we can extract a subsequence which converges weakly to some $(\bar{p}, \bar{q})$. A convex closed in $L^1(\mu) \times L^1(\nu)$ is weakly closed, so $(\bar{p}, \bar{q}) \in A$. Since $J$ is linear and continuous on $L^1(\mu) \times L^1(\nu)$, we get: $J(\bar{p}, \bar{q}) = \lim n J(p_n, q_n) = \sup(D)$. 


Proposition 2: Problem (D) has a solution \((\bar{p}, \bar{q})\) with
\[\begin{align*}
\bar{p} &= \bar{q}^\flat & 0 \leq \bar{p} \leq 1 & \bar{p}(\alpha) = 0 \\
\bar{q} &= \bar{p}^{\#} & 0 \leq \bar{q} \leq 1 & \bar{q}(\omega) = 0
\end{align*}\]
Proposition 2: Problem (D) has a solution \((\bar{p}, \bar{q})\) with
\[
\bar{p} = \bar{q}^\flat \quad 0 \leq \bar{p} \leq 1 \quad \bar{p}(\alpha) = 0
\]
\[
\bar{q} = \bar{p}^\# \quad 0 \leq \bar{q} \leq 1 \quad \bar{q}(\omega) = 0
\]

Proof: Take a maximizing sequence \((p_n, q_n) \in A\)
Proposition 2: Problem (D) has a solution \((\bar{p}, \bar{q})\) with
\[
\bar{p} = \bar{q}^\flat \quad 0 \leq \bar{p} \leq 1 \quad \bar{p}(\alpha) = 0 \\
\bar{q} = \bar{p}^\sharp \quad 0 \leq \bar{q} \leq 1 \quad \bar{q}(\omega) = 0
\]

Proof: Take a maximizing sequence \((p_n, q_n) \in A\)

- By preceding results, we may assume \(p_n = q_n^\flat\) and \(q_n = p_n^\sharp\)
- \(p_n(\alpha) = 0, \ q_n(\omega) = 0, \) and \(\inf_{y \in E^-} q_n(y) = 0\)
Proposition 2: Problem (D) has a solution \((\bar{p}, \bar{q})\) with
\[
\bar{p} = \bar{q}^\flat \quad 0 \leq \bar{p} \leq 1 \quad \bar{p}(\alpha) = 0
\]
\[
\bar{q} = \bar{p}^\sharp \quad 0 \leq \bar{q} \leq 1 \quad \bar{q}(\omega) = 0
\]

Proof: Take a maximizing sequence \((p_n, q_n) \in A\)

- By preceding results, we may assume \(p_n = q_n^\flat\) and \(q_n = p_n^\sharp\)
  \(p_n(\alpha) = 0, \ q_n(\omega) = 0, \) and \(\inf_{y \in E^-} q_n(y) = 0\)

- Then, for all \(x \in E^+, p_n(x) = \min\{1, \inf_{y \in \Gamma(x) \cap E^-} q_n(y)\}\)
Proposition 2: Problem (D) has a solution $(\bar{p}, \bar{q})$ with

\[
\begin{align*}
\bar{p} &= \bar{q}^\flat \\
0 &\leq \bar{p} \leq 1 \\
\bar{p}(\alpha) &= 0 \\
\bar{q} &= \bar{p}^\sharp \\
0 &\leq \bar{q} \leq 1 \\
\bar{q}(\omega) &= 0
\end{align*}
\]

Proof: Take a maximizing sequence $(p_n, q_n) \in A$

- By preceding results, we may assume $p_n = q_n^\flat$ and $q_n = p_n^\sharp$
  
  $p_n(\alpha) = 0$, $q_n(\omega) = 0$, and $\inf_{y \in E^-} q_n(y) = 0$

- Then, for all $x \in E^+$, $p_n(x) = \min \{1, \inf_{y \in \Gamma(x) \cap E^-} q_n(y)\}$

- This implies $0 \leq p_n(x) \leq 1$. Similarly, we get $0 \leq q_n(x) \leq 1$
A Dual Solution

**Proposition 2:** Problem (D) has a solution \((\bar{p}, \bar{q})\) with

\[
\bar{p} = \bar{q}^♭ \quad 0 \leq \bar{p} \leq 1 \quad \bar{p}(\alpha) = 0
\]

\[
\bar{q} = \bar{p}^♯ \quad 0 \leq \bar{q} \leq 1 \quad \bar{q}(\omega) = 0
\]

**Proof:** Take a maximizing sequence \((p_n, q_n) \in A\)

- By preceding results, we may assume \(p_n = q_n^♭\) and \(q_n = p_n^♯\)
  
  \(p_n(\alpha) = 0, q_n(\omega) = 0,\) and \(\inf_{y \in E^-} q_n(y) = 0\)

- Then, for all \(x \in E^+\), \(p_n(x) = \min \{1, \inf_{y \in \Gamma(x) \cap E^-} q_n(y)\}\)

- This implies \(0 \leq p_n(x) \leq 1\). Similarly, we get \(0 \leq q_n(x) \leq 1\)

- So the family \((p_n, q_n)\) is equi-integrable in \(L^1(\mu) \times L^1(\nu)\)
A Dual Solution

**Proposition 2:** *Problem (D) has a solution* \((\bar{p}, \bar{q})\) *with*

\[
\bar{p} = \bar{q}^\flat \quad 0 \leq \bar{p} \leq 1 \quad \bar{p}(\alpha) = 0
\]
\[
\bar{q} = \bar{p}^\sharp \quad 0 \leq \bar{q} \leq 1 \quad \bar{q}(\omega) = 0
\]

**Proof:** Take a maximizing sequence \((p_n, q_n) \in A\)

- By preceding results, we may assume \(p_n = q_n^\flat\) and \(q_n = p_n^\sharp\)
  \(p_n(\alpha) = 0, \quad q_n(\omega) = 0, \text{ and } \inf_{y \in E^-} q_n(y) = 0\)

- Then, for all \(x \in E^+\), \(p_n(x) = \min \{1, \inf_{y \in \Gamma(x) \cap E^-} q_n(y)\}\)

- This implies \(0 \leq p_n(x) \leq 1\). Similarly, we get \(0 \leq q_n(x) \leq 1\)

- So the family \((p_n, q_n)\) is equi-integrable in \(L^1(\mu) \times L^1(\nu)\)

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- \(\mathcal{A}\) convex closed in \(L^1(\mu) \times L^1(\nu)\) is weakly closed, so \((\bar{p}, \bar{q}) \in \mathcal{A}\)
- Since \(J\) is linear and continuous on \(L^1(\mu) \times L^1(\nu),\) we get:
  \[
  J(\bar{p}, \bar{q}) = \lim_{n} J(p_n, q_n) = \sup(D)
  \]
Introduction: Open Pit Mining

A Continuous Space Model

An Optimal Transportation Problem

The Kantorovich Dual

Elements of $c$-Convex Analysis

Solving the Dual Problem

Solving the Optimum Pit Problem

Perspectives
Complementary Slackness, and Monotonicity

If $\pi$ is optimal to problem (K) and $(p,q)$ to its dual (D), then
\[0 = J(p,q) - \int_{X \times Y} c(x,y) \, d\pi = \int_{X \times Y} (p(x) - q(y) - c(x,y)) \, d\pi\]
implies the CS conditions:
\[p(x) - q(y) - c(x,y) = 0, \quad \pi \text{-a.e.}\]

Denote $y \in \Gamma(x)$ by:
\[y \gtrless x \quad \text{(the preorder on E defined by } \Gamma)\]

Monotonicity Lemma: If $(\bar{p}, \bar{q})$ is an optimal solution to (D) satisfying the properties in Proposition 2, then
\[y'' \gtrless y' \gtrless x'' \gtrless x' \Rightarrow \bar{q}(y'') \geq \bar{q}(y') \geq \bar{p}(x'') \geq \bar{p}(x')\]

Proof: The first and last inequalities follow from $\bar{q} = \bar{p}^\#, \bar{p} = \bar{q}^\♭$, and $c$-Fenchel conjugates increasing w.r.t. the preorder $\gtrless$. The middle inequality follows from $\bar{p}^\#(y) = \max\{\bar{p}(\alpha), \sup_{x: y \in \Gamma(x)} \bar{p}(x)\}$ for all $y \in E$. ▶
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Proposition 3:

Let \((\bar{p}, \bar{q})\) be an optimal solution to problem \((D)\) satisfying the properties in Proposition 2. Then

\[ F := \{x | \bar{p}(x) = 1\} \cup \{y | \bar{q}(y) = 1\} \]

defines an optimum pit.

Proof:

Letting \(F^+ := F \cap E^+\) and \(F^- := F \cap E^-\), we have

\[ g(F^-) = \int F^- d\mu - \int F^+ d\nu \leq \sup(P) \]

Let \(G^+ := E^+ \setminus F^+\) and \(G^- := E^- \setminus F^-\):

since \(\bar{p} = 1\) on \(F^+\), \(\bar{q} = 1\) on \(F^-\), and \(\bar{p}(\alpha) = \bar{q}(\omega) = 0\),

\[ J(\bar{p}, \bar{q}) = \int F^+ d\mu - \int F^- d\nu + \int G^+ \bar{p} d\mu - \int G^- \bar{q} d\nu \]
**Proposition 3:** Let \((\bar{p}, \bar{q})\) be an optimal solution to problem (D) satisfying the properties in Proposition 2. Then

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  \[ J(\bar{p}, \bar{q}) = \int_{F^+} d\mu - \int_{F^-} d\nu + \int_{G^+} \bar{p} d\mu - \int_{G^-} \bar{q} d\nu \]
Proof, continued

\[ \text{Since } \nu \text{ is a marginal of } \pi, \int G - \bar{q}(y) \, d\nu(y) = \int E + \times G - \bar{q}(y) \, d\pi(x,y) \]

\[ c(x,y) = 0 \text{ or } +\infty \text{ for } (x,y) \in E_+ \times E_-, \text{ CS conditions,} \]

\[ 0 \leq \bar{p} \leq 1 \text{ and } 0 \leq \bar{q} \leq 1 \text{ imply that } \bar{p}(x) = \bar{q}(y) \pi\text{-a.e. on } E_+ \times E_- \text{. Thus:} \]

\[ \pi(F_+ \times G - \bar{q}(y)) = 0 = \pi(G_+ \times F - \bar{p}(x)) \] (zero allocations between excavated and unexcavated points), and

\[ \int E_+ \times G - \bar{q}(y) \, d\pi(x,y) = \int G_+ \times G - \bar{q}(y) \, d\pi(x,y) = \int G_+ \times \bar{p}(x) \, d\pi(x,y) = \int G_+ \times \bar{p}(x) \, d\mu(x) \]

\[ \Rightarrow J(\bar{p},\bar{q}) = \int F_+ \, d\mu - \int F_- \, d\nu = g(F) \]

\[ \text{Hence } g(F) = J(\bar{p},\bar{q}) = \sup(D) = \inf(K) \geq \sup(P) \geq g(F) \]
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$$\Rightarrow \quad J(\bar{p}, \bar{q}) = \int_{F^+} d\mu - \int_{F^-} d\nu = g(F)$$
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$$\implies J(\bar{p}, \bar{q}) = \int_{F^+} d\mu - \int_{F^-} d\nu = g(F)$$

- Hence $g(F) = J(\bar{p}, \bar{q}) = \sup(D) = \inf(K) \geq \sup(P) \geq g(F)$
Main Result

**Theorem:** If

- $E$ is compact,
- $\Gamma$ is reflexive, transitive and has a closed graph, and
- $g(x)$ is continuous with $\int_E \max\{0, g(x)\} \, dx > 0$,

then:

1. Problem $(P)$ has an optimum solution, i.e., an optimal pit $F$
2. Its indicator functions $(p_F, q_F)$ define optimum potentials, i.e., optimal solutions to $(D)$
3. Problem $(K)$ has an optimum solution (profit allocation) and is a strong dual to $(P)$, i.e., $\min(K) = \max(P)$
4. A pit $F$ is optimal iff there exists a feasible solution $\pi$ to $(K)$ such that $(p_F, q_F)$ satisfies the CS conditions
Theorem [Matheron, 1975; also Topkis, 1976]:

1. The family $\mathcal{F}$ of all pits is closed under arbitrary unions and intersections:

$$\bigcup_{F \in G} F \in \mathcal{F} \quad \text{and} \quad \bigcap_{F \in G} F \in \mathcal{F} \quad \text{for all } G \subseteq \mathcal{F}$$
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3. There exist a unique smallest optimum pit and a unique largest optimum pit

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- **Dynamic version**: profits in the distant future should be discounted
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Recall: production planning models include excavating and processing decisions over time, subject to capacity constraints, and with discounted cash flows

Taking uncertainties into account:
- Geological uncertainties on rock properties, amounts and location of ore, etc.
- Operational uncertainties (disruptions)
- Economic uncertainties, in particular, market prices of the minerals

Formulating a local maximum-flow, minimum-cut model (instead of the "global" transportation model) as is done for image segmentation and processing? a fluid dynamics model?

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  geological uncertainties on rock properties, amounts and location of ore, etc.
  
  operational uncertainties (disruptions)
  
  economic uncertainties, in particular, market prices of the minerals

Formulating a local maximum-flow, minimum-cut model (instead of the “global” transportation model)
  
  as is done for image segmentation and processing?
  
  a fluid dynamics model?
Perspectives...

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- Numerical *implementation*
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  - as is done for image segmentation and processing?
  - a fluid dynamics model?

- **Numerical implementation**
  - different from a blocks model...
That’s it, folks.

Any questions?