Geometric and topological characterizations of strong duality in nonconvex optimization with a single equality and geometric constraints

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Contents

Constrained Optimization

X real loc. conv. top. vec. sp., and $\emptyset \neq C \subseteq X$. Given $f : C \to \mathbb{R}$ and $g : C \to \mathbb{R}$, consider the constrained minimization problem

$$\mu \doteq \inf\{f(x) : \ g(x) = 0, \ x \in C\}. \qquad (P)$$

The Lagrangian dual problem associated to (P) is

$$\nu \doteq \sup_{\lambda^* \in \mathbb{R}} \inf_{x \in C} [f(x) + \lambda^* g(x)]. \qquad (D)$$

We say: (P) has a (Lagrangian) *zero duality gap* if $\mu = \nu$; (P) has *strong duality* if it has a zero duality gap and Problem (D) admits a solution.

A continuous-version of SQP

$$\mu_q \doteq \min\Big\{ f(x) \doteq \frac{1}{2}\int_0^1 x^\top(t)Ax(t)dt : \ g(x) \doteq \int_0^1 e^\top(t)x(t)dt - 1 = 0,$$

$$x \in C \doteq L^2_+(]0,1[;\mathbb{R}^n) \Big\}.$$

Here, $A = (a_{ij})$ is a real symmetric copositive matrix, i. e., $x^\top Ax \geq 0$ for all $x \in \mathbb{R}^n_+$; $e \in \mathrm{qi}\ L^2_+(]0,1[;\mathbb{R}^n) = L^2_{++}(]0,1[;\mathbb{R}^n)$. It is known $\{x \in L^2_+(]0,1[;\mathbb{R}^n) : \ \langle e, x\rangle = 1\}$ is a weakly compact base of $L^2_+(]0,1[;\mathbb{R}^n)$. Thus, the dual is

$$\sup_{\lambda\in\mathbb{R}} \inf_{x\in L^2_+} L(\lambda, x) = \frac{1}{2}\int_0^1 x(t)^\top Ax(t) + \lambda\Big(\int_0^1 e(t)^\top x(t)dt - 1\Big). \tag{1}$$

Introduce, as usual, the Lagrangian

$$L(\gamma, \lambda, x) = \gamma f(x) + \lambda g(x), \ \ \gamma \geq 0, \ \lambda \in \mathbb{R}.$$

By setting $K \doteq \{x \in C : \ g(x) = 0\}$, we obtain (weak duality)

$$\inf_{x \in C} L(\gamma, \lambda, x) \leq \inf_{x \in K} L(\gamma, \lambda, x) \leq \gamma \inf_{x \in K} f(x), \quad \forall \, \gamma \geq 0, \, \forall \, \lambda \in \mathbb{R}.$$

In order to get the equality, we need to find conditions under which the reverse inequality holds, that is, we must have:

$$\gamma(f(x) - \mu) + \lambda g(x) \geq 0 \quad \forall \, x \in C. \tag{2}$$

This will imply strong duality once we get $\gamma > 0$. Denote $F \doteq (f, g)$, the sets

$$\mathcal{F} \doteq F(C) + \mathbb{R}_+(1, 0), \ \mathcal{F}_\mu \doteq \mathcal{F} - \mu(1, 0), \tag{3}$$

will play an important role in our analysis.

Then,

$$(\gamma,\lambda)\in[\overline{\overline{\text{cone}}\ \mathcal{F}_\mu}]^* = [\overline{\text{cone}}\ \mathcal{F}_\mu]^* = [\text{cone}\ \mathcal{F}_\mu]^* = [\mathcal{F}_\mu]^*. \qquad (4)$$

Set

$$\mathcal{L}_{SD} \doteq \Big\{\lambda\in\mathbb{R}:\ (1,\lambda)\in[\text{cone}\ \mathcal{F}_\mu]^*\Big\}. \qquad (5)$$

Then, (P) has SD property if, and only if $\mathcal{L}_{SD}\neq\emptyset$. Hence

$$\mathcal{L}_{SD}\subseteq\mathcal{S}_D,$$

where $\mathcal{S}_D$ is the solution set to the dual problem (D).

Furthermore, we need the following numbers:

- if $\Omega_+^- \doteq S_f^-(\mu) \cap S_g^+(0) \neq \emptyset$,

$$s \doteq \sup_{x \in \Omega_+^-} \frac{g(x)}{f(x) - \mu} \ \in \ ]-\infty, 0];$$

- if $\Omega_-^- \doteq S_f^-(\mu) \cap S_g^-(0) \neq \emptyset$,

$$l \doteq \inf_{x \in \Omega_-^-} \frac{g(x)}{f(x) - \mu} \ \in \ [0, +\infty[;$$

The geometric and topological characterizations of SD:

Theorem: [Cárcamo-FB, 2015]

Consider problem (P) with $\mu \in \mathbb{R}$. Then, (a), (b) and (c) are equivalent:

(a) Strong Duality holds for (P), that is

$$\exists\, \lambda_0^* \in \mathbb{R}: \ f(x) + \lambda_0^* g(x) \geq \mu, \ \forall\, x \in C; \tag{6}$$

(b) $\overline{\mathrm{cone}}(\mathcal{F}_\mu) \cap (-\mathbb{R}_{++} \times \{0\}) = \emptyset$ and $\overline{\mathrm{cone}}(\mathcal{F}_\mu)$ is convex;

(c) $\mathrm{cone}(\mathcal{F}_\mu)$ is convex and exactly one of the following assertions holds:

- $(c1)$ $S_f^-(\mu) = \emptyset$, in which case $0 \in \mathcal{L}_{SD}$;
- $(c2)$ $\Omega_+^- \neq \emptyset, \ s < 0$, in which case $\min \mathcal{L}_{SD} = -\dfrac{1}{s}$;
- $(c3)$ $\Omega_-^- \neq \emptyset, \ l > 0$, in which case $\max \mathcal{L}_{SD} = -\dfrac{1}{l}$.

Theorem (continued ...)

Consequently, under condition (a), one obtains

$$\inf_{x\in K} f(x) = \inf_{\substack{\lambda_0^* g(x)\leq 0 \\ x\in C}} f(x); \tag{7}$$

$$\bar{x} \text{ is a solution to } (P) \Longleftrightarrow \begin{cases} \bar{x}\in C, \quad g(\bar{x})=0, \\ f(\bar{x}) = \inf_{x\in C}[f(x)+\lambda_0^* g(x)] \end{cases} \tag{8}$$

and $\mathcal{L}_{SD} = \mathcal{S}_D$.

Remark

We point out that the convexity of $\overline{\mathrm{cone}}(\mathcal{F}_\mu)$ does imply the convexity of $\mathrm{cone}(\mathcal{F}_\mu)$ without SD. This is illustrated by the functions $f(x_1, x_2) = 2x_1x_2$, $g(x_1, x_2) = x_1$ and $C = \mathbb{R}^2$. Then, $\mu = 0$, $F(\mathbb{R}^2) = \{(0,0)\} \cup (\mathbb{R}^2 \setminus \mathbb{R} \times \{0\})$, and so

$$\mathrm{cone}(\mathcal{F}_\mu) = \mathbb{R}^2 \setminus (-\mathbb{R}_{++} \times \{0\}),$$

which is nonconvex, but $\overline{\mathrm{cone}}(\mathcal{F}_\mu) = \mathbb{R}^2$.

The following result, which is new in the literature, provides a characterization of strong duality under a Slater-type condition.

Corollary: [Cárcamo-FB, 2015]

Let $\mu \in \mathbb{R}$ and assume that there exist $x_1, x_2 \in C$ such that $g(x_1) < 0 < g(x_2)$. Then, $\mathrm{cone}(\mathcal{F}_\mu)$ is convex if, and only if strong duality holds for (P).

The case f and g quadratic: $C = \mathbb{R}^n$; $F = (f, g)$:

Corollary [Opazo-FB, 2014]: Let $\mu \in \mathbb{R}$

Assume that there exist $x_1, x_2 \in \mathbb{R}^n$ st $g(x_1) < 0 < g(x_2)$. Then, $F(\mathbb{R}^n) + \mathbb{R}_+(1, 0)$ is convex if, and only if SD holds.

Lemma [Opazo-FB, 2014]:

$F(\mathbb{R}^n) + \mathbb{R}_+(1, 0)$ is convex if, and only if any of the following conditions is satisfied:

$(C1)$ $F_L(\ker A \cap \ker B) \neq \{0\}$; $F_L(u) = (\langle a, u \rangle, \langle b, u \rangle)$;

$(C2)$ $B \neq 0$;

$(C3)$ $u \in \mathbb{R}^n, \ \langle Bu, u \rangle = 0 \Longrightarrow \langle Au, u \rangle \geq 0$;

$(C4)$ $\exists \, u \in \mathbb{R}^n, \ \langle Au, u \rangle < 0, \ \langle Bu, u \rangle = 0, \ \langle b, u \rangle = 0$.

This characterization encompasses the case when the Hessian of g is non-null, or when g is strictly concave (or convex).

Corollary [Opazo-FB, 2014]:

$F(\mathbb{R}^n) + P$ is convex for all convex cone $P \subseteq \mathbb{R}^2$ with $\operatorname{int} P \neq \emptyset$.

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KKT optimality conditions

This section deals with some characterizations of the validity of the KKT optimality conditions for the problem (P). For simplicity, take $X = \mathbb{R}^n$, and f and g to be Gâteaux differentiable on $\mathbb{R}^n$. Such characterizations will be derived as a consequence of our main theorem on SD applied to the linearized approximation problem defined, given $\bar{x} \in C$, by

$$\mu_L \doteq \inf_{v \in G'(\bar{x})} \nabla f(\bar{x})^\top v, \tag{9}$$

where

$$G'(\bar{x}) \doteq \Big\{ v \in T(C; \bar{x}) : \ \nabla g(\bar{x})^\top v = 0 \Big\}.$$

Here, $T(C; \bar{x})$ stands for the contingent cone of C (or tangent cone of Bouligand) at $\bar{x}$, which is always a closed cone. Set $F_L(v) \doteq (\nabla f(\bar{x})^\top v, \nabla g(\bar{x})^\top v)$. It is obvious that $\mu_L \in \{-\infty, 0\}$.

In view of Theorem 8, we introduce the following sets:

$$\widehat{S}_f^-(0) \doteq \{v \in T(C;\bar{x}) : \nabla f(\bar{x})^\top v < 0\},$$

$$\widehat{S}_g^+(0) \doteq \{v \in T(C;\bar{x}) : \nabla g(\bar{x})^\top v > 0\},$$

$$\widehat{\Omega}_+^- \doteq \widehat{S}_f^-(0) \cap \widehat{S}_g^+(0), \quad \widehat{\Omega}_-^- \doteq \widehat{S}_f^-(0) \cap \widehat{S}_g^-(0).$$

Furthermore, whenever $\widehat{\Omega}_+^- \neq \emptyset \neq \widehat{\Omega}_-^-$, we put

$$\widehat{s} \doteq \sup_{v \in \widehat{\Omega}_+^-} \frac{\nabla g(\bar{x})^\top v}{\nabla f(\bar{x})^\top v}, \quad \widehat{l} \doteq \inf_{v \in \widehat{\Omega}_-^-} \frac{\nabla g(\bar{x})^\top v}{\nabla f(\bar{x})^\top v}.$$

Denote by $\mathcal{L}(\bar{x})$ the set of Lagrange multipliers to (P) associated to a (not necessarily feasible) point $\bar{x} \in C$, i. e., the set of $\lambda^* \in \mathbb{R}$ satisfying (10). When $\mathcal{L}(\bar{x}) \neq \emptyset$, we say that $\bar{x}$ is a KKT point.

Let $\bar{x} \in C$. In case $\nabla g(\bar{x}) = 0$, it is not difficult to check that:

- $\mu_L = 0$ if, and only if $\mathcal{L}(\bar{x}) = \mathbb{R}$.
- $\mu_L = -\infty$ if, and only if $\mathcal{L}(\bar{x}) = \emptyset$.

Theorem: [Cárcamo-FB, 2015]

Assume that $\bar{x} \in C$. The following assertions are equivalent:

(*a*) $\exists\ \lambda^* \in \mathbb{R}$ such that

$$\nabla f(\bar{x}) + \lambda^* \nabla g(\bar{x}) \in [T(C;\bar{x})]^*. \tag{10}$$

(*b*) $\mu_L = 0$ and strong duality holds for the problem (9).

(*c*) $\overline{F_L(T(C;\bar{x})) + \mathbb{R}_+(1,0)}$ is convex and

$$\overline{[F_L(T(C;\bar{x})) + \mathbb{R}_+(1,0)]} \cap (-\mathbb{R}_{++} \times \{0\}) = \emptyset. \tag{11}$$

Theorem (continued ...)

(*d*) $F_L(T(C;\bar{x})) + \mathbb{R}_+(1,0)$ is convex and exactly one of the following assertions holds:

- (*d*1) $\widehat{S}_f^-(0) = \emptyset$, in which case $0 \in \mathcal{L}(\bar{x})$;
- (*d*2) $\widehat{\Omega}_+^- \neq \emptyset$, $\widehat{s} < 0$, in which case $\min \mathcal{L}(\bar{x}) = -\dfrac{1}{\widehat{s}}$;
- (*d*3) $\widehat{\Omega}_-^- \neq \emptyset$, $\widehat{l} > 0$, in which case $\max \mathcal{L}(\bar{x}) = -\dfrac{1}{\widehat{l}}$.

(*e*) $\overline{F_L(T(C;\bar{x})) + \mathbb{R}_+(1,0)}$ is convex, $\mu_L = 0$ and

$$\left.\begin{array}{l} v_k \in T(C;\bar{x}), \|v_k\| \to +\infty, \\ \nabla g(\bar{x})^\top v_k \to 0, \nabla f(\bar{x})^\top v_k < 0] \end{array}\right\} \Longrightarrow \overline{\lim_k}\, \nabla f(\bar{x})^\top v_k = 0. \tag{12}$$

A simple sufficient condition for a minimum to be a KKT point, under strong duality is expressed in the following result.

Proposition [Cárcamo-FB, 2015]:

Assume that strong duality holds for (P). Then, every solution to (P) is a KKT point, that is, $\mathcal{L}_{SD} \subseteq \mathcal{L}(\bar{x})$ for all $\bar{x} \in \operatorname*{argmin}_K f$.

It may applied to situations where results based either on exact penalization techniques ([Yang-Peng, MOR 2007]) or where Abadie's constraint qualification fail. In addition, there are instances where no minimizer is a KKT point, if strong duality is not satisfied. For 1st case:

$$0 = \mu \doteq \min\{f(x_1, x_2) \doteq x_2 : g(x_1, x_2) \doteq x_2 - x_1^2 = 0, (x_1, x_2) \in \mathbb{R}^2\}.$$

For 2nd case: $f(x_1, x_2) = x_2$, $g(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2 - 1$ and $C \doteq \{(x_1, x_2) \in \mathbb{R}^2 : g_0(x_1, x_2) \leq 0\}$ with $g_0(x_1, x_2) \doteq (x_1 - 1)^2 + (x_2 + 1)^2 - 1$.

Nonconvex QP with two quadratic equality constraints

We now discuss the problem:

$$\mu \doteq \min\{f(x) : \; g_1(x) = 0, \; g_2(x) = 0\}, \tag{13}$$

where we specialize the functions f, g_i, $i = 1, 2$ to be (non necessarily homogeneous) quadratic. Here,
$C \doteq \{x \in \mathbb{R}^n : \; g_2(x) = 0\}$, $K \doteq \{x \in C : \; g_1(x) = 0\}$,

$$f(x) \doteq \frac{1}{2}x^\top Ax + a^\top x + \alpha, \;\; g_i(x) \doteq \frac{1}{2}x^\top B_i x + b_i^\top x + \beta_i, \;\; i = 1, 2,$$

with $A = A^\top$, $B_i = B_i^\top$; $a, b_i \in \mathbb{R}^n$ and α, β_i being real numbers.
In addition to the dual problem

$$\nu \doteq \sup_{\lambda_1 \in \mathbb{R}} \inf_{x \in C} \{f(x) + \lambda_1 g_1(x)\}, \tag{14}$$

consider also the standard (Lagrangian) dual problem to (13):

$$\nu_0 \doteq \sup_{\lambda_1,\lambda_2 \in \mathbb{R}} \inf_{x \in \mathbb{R}^n} \{f(x) + \lambda_1 g_1(x) + \lambda_2 g_2(x)\}. \tag{15}$$

We say that problem (13) has standard strong duality (SSD) if $\mu = \nu_0$ and problem (15) admits solution. It is easy to check that

$$\nu_0 \leq \nu \leq \mu.$$

One the other hand, given a feasible point $\bar{x}$, it is said that $\bar{x}$ is a standard KKT point to problem (13), if for some $\lambda_1, \lambda_2 \in \mathbb{R}$, one has

$$\nabla f(\bar{x}) + \lambda_1 \nabla g_1(\bar{x}) + \lambda_2 \nabla g_2(\bar{x}) = 0.$$

Set

$$Z(\bar{x}) \doteq \{v \in \mathbb{R}^n : \ \nabla g_i(\bar{x})^\top v + \frac{1}{2} v^\top B_i v = 0, \ \ i = 1, 2\}.$$

It is known that

$$T(C; \bar{x}) = \left\{ v \in \mathbb{R}^n : \ \nabla g_2(\bar{x})^\top v = 0 \ \right\} = \nabla g_2(\bar{x})^\perp \ \text{if}\ \nabla g_2(\bar{x}) \neq 0,$$

and so $[T(C; \bar{x})]^* = \mathbb{R} \nabla g_2(\bar{x})$; whereas

$$T(C; \bar{x}) = \left\{ v \in \mathbb{R}^n : \ v^\top B_2 v = 0 \ \right\} \ \text{if} \ \ \nabla g_2(\bar{x}) = 0.$$

The latter set is, in general, nonconvex. However, in case B_2 is positive semidefinite, or equivalently, g_2 is convex (for instance, when such an equality constraint corresponds to a component of x taking the value either 0 or 1), with $\nabla g_2(\bar{x}) = 0$, we obtain $T(C; \bar{x}) = \ker B_2$, and so $[T(C; \bar{x})]^* = (\ker B_2)^\perp = B_2(\mathbb{R}^n)$.

Next theorem, which is new, provides 1st and 2nd order necessary optimality conditions under additional assumptions besides SD. It proves that every optimal solution is a standard KKT point.

Theorem [Cárcamo-FB, 2015]: Let $\mu \in \mathbb{R}$

Let f, g_1, g_2 be quadratic, $\bar{x}$ feasible satisfying $\nabla g_2(\bar{x}) \neq 0$. Set $C = \{x \in \mathbb{R}^n : \; g_2(x) = 0\}$. Then $(a) \Longrightarrow (b)$, where

- (a) $\bar{x}$ is a solution to (13) and SD holds;
- (b) $\exists\ \lambda_1, \lambda_2 \in \mathbb{R}$ such that $\nabla f(\bar{x}) + \lambda_1 \nabla g_1(\bar{x}) + \lambda_2 \nabla g_2(\bar{x}) = 0$, $A + \lambda_1 B_1 + \lambda_2 B_2 \succcurlyeq 0$ on $Z_2(\bar{x}) \cup \nabla g_2(\bar{x})^{\perp}$.

It may be applied to instances without satisfying Abadie's CQ.

$$Z_2(\bar{x}) \doteq \{v \in \mathbb{R}^n : \; \nabla g_2(\bar{x})^\top v + \frac{1}{2} v^\top B_2 v = 0\}.$$

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A concrete application

$$\mu_q \doteq \min\Big\{ f(x) \doteq \frac{1}{2}\int_0^1 x^\top(t)Ax(t)dt : \ g(x) \doteq \int_0^1 e^\top(t)x(t)dt - 1 = 0,$$

$$x \in C \doteq L^2_+(]0,1[;\mathbb{R}^n)\Big\}.$$

Here, $e \in \mathrm{qi}\, L^2_+(]0,1[;\mathbb{R}^n) = L^2_{++}(]0,1[;\mathbb{R}^n)$. $F = (f,g)$.

Proposition: Assume $\mu_q > 0$,

(a) $\Omega^-_+ = \Omega^=_+ = \emptyset$, and therefore $S^+_g(0) = \Omega^+_+ \neq \emptyset$;

(b) $\emptyset \neq S^-_f(\mu_q) = \Omega^-_-$;

(c) $m = l = \dfrac{1}{2\mu_q}$, so $l > 0$ and $\mathcal{L}_{SD} = \mathcal{S}_D = \{-2\mu_q\}$, and so
$$\mathrm{cone}(F(C) + \mathbb{R}_+(1,0) - \mu_q(1,0)) = \Big\{(u,v) : v \leq \frac{1}{2\mu_q}u\Big\};$$

(d) strong duality holds.

By Lyapunov theorem $F(C) + \mathbb{R}_+(1,0)$ is convex.

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