Pointwise Second Order Optimality Conditions in Optimal Control

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In collaboration with D. Hoehener

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Diary of Nice Days

- 1980-1981 PhD student at SISSA
- May 1981 - met Terry Rockafellar at IIASA for one day. (Cellina's idea) First discussion. **Thank you, Terry.**
- May 1984 - A recommendation letter by Terry for a CR position at CNRS. Got this first position. **Thank you, Terry.**
- December 1984 - Terry reported on my Doctorat d'Etat (habilitation) and came for habilitation. **Thank you, Terry.**
- May 1986 - First stay in Terry's home. **Thank you, Terry.**

Outline of the talk

Mayer's Optimal Control Problem

Consider the minimization problem

$$\text{Minimize } \varphi(x(1)) \qquad \text{(P)}$$

over absolutely continuous $x \in W^{1,1}([0,1];\mathbb{R}^n)$ satisfying

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) & \text{a.e. in } [0,1] \\ u(t) \in U(t) & \text{a.e. in } [0,1] \\ x(0) \in K_0 \end{cases}$$

where $\varphi\colon \mathbb{R}^n \to \mathbb{R}$, $f\colon [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $U\colon [0,1] \rightsquigarrow \mathbb{R}^m$ is a measurable set-valued map with closed nonempty images and $K_0 \subset \mathbb{R}^n$ is closed.
Let $\bar{x}$ be a **minimizer** and $\bar{u}$ be a corresponding **control**.

Standard Assumptions (SA)

- For all $(t, x, u) \in [0, 1] \times \mathbb{R}^n \times \mathbb{R}^m$, $f(\cdot, x, u)$ is measurable, $f(t, x, \cdot)$ is continuous and there exists $a_1 > 0$ such that $\sup_{u \in U(t)} |f(t, x, u)| \leq a_1(|x| + 1)$;
- $\forall\ R > 0$, there exists an integrable map $k_R : [0, 1] \to \mathbb{R}_+$ such that for a.e. $t \in [0, 1]$ and all $u \in U(t)$

$$|f(t, x, u) - f(t, y, u)| \leq k_R(t) \, |x - y| \, , \ \forall x, \ y \in RB;$$

- $\exists\ a_2 > 0, \ \rho > 0, \ f(t, \cdot, u) \in C^2$ on $\bar{x}(t) + \rho B$, $\sup_{u \in U(t)} \|f_x(t, \bar{x}(t), u)\| \leq a_2$ for a.e. $t \in [0, 1]$;
- $\exists\ \ell \in L^1([0, 1]; \mathbb{R}_+)$, s.t. for a.e. $t \in [0, 1]$, $\forall\ u \in U(t)$ $\|f_x(t, x, u) - f_x(t, y, u)\| \leq \ell(t) \, |x - y|$, $\forall\ x, y \in \bar{x}(t) + \rho B$;
- φ is differentiable

First Order Necessary Conditions

The **Hamiltonian** $\mathcal{H}\colon [0,1]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$ is defined by

$$\mathcal{H}(t,x,p,u)=\langle p, f(t,x,u)\rangle$$

The maximum principle states that the solution $\bar{p}\in W^{1,1}([0,1];\mathbb{R}^n)$ of the **adjoint system**

$$\begin{cases} -\dot{p}(t)=\mathcal{H}_x(t,\bar{x}(t),p(t),\bar{u}(t)) & \text{a.e.}\\ -p(1)=\nabla\varphi(\bar{x}(1)) \end{cases}$$

satisfies the **transversality** condition

$$\bar{p}(0)\in N^{\flat}_{K_0}(\bar{x}(0))$$

and the **maximality** condition for a.e. $t\in[0,1]$:

$$\max_{u\in U(t)}\langle \bar{p}(t), f(t,\bar{x}(t),u)-f(t,\bar{x}(t),\bar{u}(t))\rangle=0$$

The maximum is attained by $\bar{u}(t)$. (Pontryagin and al.)

Second Order Tangents

Adjacent tangent to $K \subset \mathbb{R}^n$ at $x \in K$ is

$$T_K^\flat(x) := \{u \in \mathbb{R}^n \mid \lim_{h \to 0+} \frac{\mathrm{dist}_K(x+hu)}{h} = 0\}$$

Second order adjacent set to K at x in the direction $u \in \mathbb{R}^n$

$$T_K^{\flat(2)}(x;u) := \{v \in \mathbb{R}^n \mid \lim_{h \to 0+} \frac{\mathrm{dist}_K(x+hu+h^2v)}{h^2} = 0\}$$

$v \in T_K^{\flat(2)}(x;u)$ if and only if $\exists\, \psi : [0,\tau) \to K$ such that

$$\psi(0) = x, \quad \dot{\psi}(0) = u, \quad \ddot{\psi}(0) = 2v$$

$\psi(\cdot)$ is Rockafellar's second order inner arc
(goes back to Ben Tal and Zowe 1980.
See also R. Cominetti 1990 and Aubin-HF 1990.)

Second Order Normals

Weak point of this notion -

$$\{(u,v) \mid v \in T_K^{\flat(2)}(x;u)\} \text{ is not closed.}$$

First order normal cone $N_K^{\flat}(x) := \left[T_K^{\flat}(x)\right]^-$

We associate with every $q \in N_K^{\flat}(x)$ the **second order "normals"**

$$N_K^{\flat(2)}(x;q) := \{Q \in \mathbf{S}(\mathrm{n}) \mid \langle q, w\rangle + \frac{1}{2}Qyy \leq 0,$$
$$\forall y \in T_K^{\flat}(x) \cap \{q\}^{\perp},\ \forall w \in T_K^{\flat(2)}(x;y)\}$$

where $\mathbf{S}(\mathrm{n})$ denotes the set of symmetric $n \times n$ matrices.

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Maximizing Controls

The set of **maximizing controls** at $t \in [0,1]$ is defined by

$$\overline{U}(t) := \arg\max_{u \in U(t)} \mathcal{H}(t, \bar{x}(t), \bar{p}(t), u)$$

Then $\bar{u}(t) \in \overline{U}(t)$ a.e.

We abbreviate $[t] := (t, \bar{x}(t), \bar{p}(t), \bar{u}(t))$,

$$\Delta f(t, \bar{x}(t), u) := f(t, \bar{x}(t), u) - f(t, \bar{x}(t), \bar{u}(t))$$

$$\Delta f_x(t, \bar{x}(t), u) := f_x(t, \bar{x}(t), u) - f_x(t, \bar{x}(t), \bar{u}(t))$$

Define

$$\overline{F}(t) := \mathrm{co}\,\{(\Delta f(t, \bar{x}(t), u), \Delta f_x(t, \bar{x}(t), u)) \mid u \in \overline{U}(t)\}$$

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Second Order Maximum Principle

Theorem

Assume φ is twice differentiable. Then the matrix solution W of the second order adjoint system

$$\begin{cases} \dot{W}(t) = -\mathcal{H}_{px}[t]W(t) - W(t)\mathcal{H}_{xp}[t] - \mathcal{H}_{xx}[t] \\ W(1) = -\varphi''(\bar{x}(1)) \end{cases}$$

satisfies the second order transversality condition

$$W(0) \in N^{\flat(2)}_{K_0}(\bar{x}(0); \bar{p}(0))$$

and the second order maximality condition

$$\max_{(v,M)\in\overline{F}(t)} \left\langle M^T\bar{p}(t) + W(t)v, v\right\rangle = 0, \quad a.e.\ in\ [0,1]$$

Additional Regularity Assumptions (ARA)

This is similar to the first order maximum principle :

$$\max_{u \in U(t)} \langle \bar{p}(t), \Delta f(t, \bar{x}(t), u) \rangle = 0 \quad \Leftrightarrow \quad \max_{v \in co\, \Delta f(t, \bar{x}(t), U(t))} \langle \bar{p}(t), v \rangle = 0$$

- For a.e. $t \in [0,1]$, $f(t,\cdot,\cdot) \in C^2$ on $(\bar{x}(t) + \rho B) \times (\bar{u}(t) + \rho B)$;
- $\exists\, a_3 > 0$ and integrable $\ell_2 \colon [0,1] \to \mathbb{R}_+$ s.t. for a.e. $t \in [0,1]$, $\|f_u(t, \bar{x}(t), \bar{u}(t))\| \leq a_3$, $\forall\, x, y \in \bar{x}(t) + \rho B$, $\forall\, u, v \in \bar{u}(t) + \rho B$

$$\|f'(t,x,u) - f'(t,y,v)\| \leq a_3(|x-y| + |u-v|)$$
$$\|f''(t,x,u) - f''(t,y,v)\| \leq \ell_2(t)(|x-y| + |u-v|)$$

$f'(t,x,u)$ and $f''(t,x,u)$ denote the Jacobian, resp. Hessian, of $f(t,\cdot,\cdot)$

Define the set of non-singular times $A := \{t \in [0,1] \mid \mathcal{H}_u[t] \neq 0\}$

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$$\|f'(t,x,u) - f'(t,y,v)\| \leq a_3(|x-y| + |u-v|)$$
$$\|f''(t,x,u) - f''(t,y,v)\| \leq \ell_2(t)(|x-y| + |u-v|)$$

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Define the set of non-singular times $A := \{t \in [0,1] \mid \mathcal{H}_u[t] \neq 0\}$

Necessary Condition for Partially Singular Controls

Theorem

Assume (SA), (ARA) and let W be the solution of the second order adjoint system. Then for a.e. $t \in [0,1]$ and for every $u \in T^{\flat}_{U(t)}(\bar{u}(t))$ satisfying one of the following conditions

(i) $t \in A$, $\mathcal{H}_u[t]u = 0$ *and* $\mathcal{H}_u[t]v + \frac{1}{2}\mathcal{H}_{uu}[t]uu = 0$ *for some* $v \in T^{\flat(2)}_{U(t)}(\bar{u}(t); u)$

(ii) $t \in [0,1] \setminus A$ *and* $\mathcal{H}_{uu}[t]uu = 0$

the Jacobson inequality holds true

$$f_u[t]^T \left(\mathcal{H}_{ux}[t] + W(t)f_u[t]\right) uu \leq 0.$$

Example: Controls Given by Inequality Constraints

Assume (SA), (ARA) and that

$$U(t) := \bigcap_{j=1}^{s} \left\{ u \in \mathbb{R}^m \;\middle|\; c^j(t,u) \leq 0 \right\}$$

where $c^j : [0,1] \times \mathbb{R}^m \to \mathbb{R}$ are measurable in t, $c^j(t,\cdot) \in C^2$.

$$I(t) := \{ j \mid c^j(t, \bar{u}(t)) = 0 \}$$

Assume $\{\nabla_u c^j(t, \bar{u}(t))\}_{j \in I(t)}$ are linearly independent for a.e. t. Then

$$T^{\flat}_{U(t)}(\bar{u}(t)) = \{ u \in \mathbb{R}^m \mid \langle \nabla_u c_j(t, \bar{u}(t)), u \rangle \leq 0 \;\; \forall\, j \in I(t) \}$$

Second Order Condition via Lagrange Multipliers

Corollary

Let W be as in the second order maximum principle. Then there exist measurable, uniquely defined (up to a set of measure zero) $\alpha_j : [0,1] \to \mathbb{R}_+$, $j = 1, ..., r$ such that for a.e. $t \in [0,1]$

(i) $\alpha_j(t) c^j(t, \bar{u}(t)) = 0$ *for all* $j \in \{1, \dots, s\}$;

(ii) $\mathcal{H}_u[t] = \sum_{j=1}^{s} \alpha_j(t) \nabla_u c^j(t, \bar{u}(t))$;

(iii) $\max_{u \in U_0(t)} f_u[t]^T \left(\mathcal{H}_{ux}[t] + W(t) f_u[t]\right) uu = 0$,

$$U_0(t) := \{u \in T^\flat_{U(t)}(\bar{u}(t)) \mid \mathcal{H}_u[t]u = 0 \ \text{ and }$$

$$(\mathcal{H}_{uu}[t] - \sum_{j=1}^{s} \alpha_j(t) c^j_{uu}(t, \bar{u}(t)))uu = 0\}.$$

Second Order Jets

Let $f\colon \mathbb{R}^n \to [-\infty, \infty]$ and $x \in dom(f)$. A pair $(q, Q) \in \mathbb{R}^n \times \mathbf{S}(\mathrm{n})$ is a **superjet** of f at x if for some $\delta > 0$ and for all $y \in x + \delta B$,

$$f(y) \leq f(x) + \langle q, y - x \rangle + \frac{1}{2} Q(y - x)(y - x) + o(|y - x|^2)$$

The set of all superjets of f at x is denoted by $J^{2,+} f(x)$.
Similarly, $(q, Q) \in \mathbb{R}^n \times \mathbf{S}(\mathrm{n})$ is a **subjet** of f at x if the above holds with $\leq$ replaced by $\geq$
The set of all subjets of f at x is denoted by $J^{2,-} f(x)$.

The value function $V\colon [0, 1] \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$V(t, y) := \inf \left\{ \varphi(z(1)) \;\middle|\; z \in \mathcal{S}_{[t,1]}(y) \right\},$$

where $y \in \mathbb{R}^n$ and $\mathcal{S}_{[t,1]}(y)$ denotes the set of solutions of the control system satisfying $x(t) = y$ and defined on $[t, 1]$.

Backward Propagation of Superjets

Theorem

Assume (SA) and that a process $(\bar{x}, \bar{u})$ *satisfies*

$$V(t_0, \bar{x}(t_0)) = \varphi(\bar{x}(1))$$

Let $\Psi \in \boldsymbol{S}(\mathrm{n})$ *be so that* $(\nabla\varphi(\bar{x}(1)), \Psi) \in J^{2,+}\varphi(\bar{x}(1))$*.*
Consider solutions $\bar{p}$ *and* W *of first and second order adjoint systems with* $\bar{p}(1) = -\nabla\varphi(\bar{x}(1))$ *and* $W(1) = -\Psi$*.*
Then W *satisfies the maximality condition a.e. in* $[t_0, 1]$ *and*

$$(-\bar{p}(t), -W(t)) \in J_x^{2,+} V(t, \bar{x}(t)), \ \forall\, t \in [t_0, 1].$$

Forward Propagation of Subjets

Theorem

Assume (SA) and that a process $(\bar{x}, \bar{u})$ *satisfies*

$$V(t_0, \bar{x}(t_0)) = \varphi(\bar{x}(1))$$

Consider the adjoint state $\bar{p}$ *defined on* $[t_0, 1]$ *and assume that for some* $W_0 \in \boldsymbol{S}(\mathrm{n})$ *we have* $(-\bar{p}(t_0), -W_0) \in J_x^{2,-}V(t_0, x_0)$*.*
Then for the solution W *of the linear matrix equation*

$$\dot{W}(t) = -\mathcal{H}_{px}[t]W(t) - W(t)\mathcal{H}_{xp}[t] - \mathcal{H}_{xx}[t], \quad W(t_0) = W_0$$

the following sensitivity relation holds true:

$$(-\bar{p}(t), -W(t)) \in J_x^{2,-}V(t, \bar{x}(t)), \ \forall\, t \in [t_0, 1].$$

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Merci pour votre attention

Critical Cone of Tangent Controls

Let $(\bar{x}, \bar{u})$ be a strong local minimizer and $\bar{p}$ be the corresponding adjoint state. Define the **local critical cone** at $\bar{u} \in \mathcal{U}$ by

$$\mathcal{C}_{loc}(\bar{u}) := \{u(\cdot) \in L^1 \mid u(t) \in T^\flat_{U(t)}(\bar{u}(t)) \text{ and } \mathcal{H}_u[t]u(t) = 0 \text{ a.e. } \}$$

and let

$$M^{(2)}(\bar{u}) := \{(u, v) \in L^\infty | u \in \mathcal{C}_{loc}(\bar{u}), v(t) \in T_U^{\flat(2)}(\bar{u}(t); u(t)) \text{ a.e. in } A\}$$

Consider the **linearized system**

$$\begin{cases} \dot{y}(t) = f_x[t]y(t) + f_u[t]u(t) & \text{a.e. in } [0,1] \\ y(0) = y_0, \end{cases}$$

where $f_x[t] := f_x(t, \bar{x}(t), \bar{u}(t))$ and $f_u[t]$ is defined in a similar way.

Quadratic functional Φ

$\forall\ (u(\cdot), v(\cdot)) \in L^2([0,1];\mathbb{R}^m) \times L^1([0,1];\mathbb{R}^m),\ y_0,\ w_0 \in \mathbb{R}^n,$

$$\Phi(u, v, y_0, w_0) := \langle -\bar{p}(0), w_0 \rangle + \frac{1}{2}\varphi''(\bar{x}(1))y(1)y(1) -$$

$$\int_0^1 (\mathcal{H}_u[t]v(t) + \frac{1}{2}\mathcal{H}_{xx}[t]y(t)y(t) + \mathcal{H}_{xu}[t]y(t)u(t) + \frac{1}{2}\mathcal{H}_{uu}[t]u(t)u(t))dt$$

where $y(\cdot)$ is the solution of the linearized system for $y_0,\ u$.

Theorem (D. Hoehener, 2012; HF and N. Osmolovskii 2015 under weaker assumptions)

If (SA), (ARA) hold true, then

$$\Phi(u, v, 0, 0) \geq 0 \quad \textit{for all} \quad (u, v) \in M^{(2)}(\bar{u}).$$

Integral Necessary Conditions

The above theorem does not allow to deduce pointwise conditions except under strong assumption on $U(t)$ because the sets

$$\{(u,v) \mid u \in T^{\flat}_{U(t)}(\bar{u}(t)),\ v \in T^{\flat(2)}_{U}(\bar{u}(t);u)\}$$

are not closed, in general. Define the **second order jets**

$$J^2_{K_0}(\bar{x}(0)) \ := \ \Big\{(y,w) \in \mathbb{R}^{2n} \ \Big| \ \forall\, h_i \to 0+,\ \exists\,(y_i,w_i) \to (y,w), \\ \langle \bar{p}(0), y_i\rangle = 0,\ \bar{x}(0) + h_i y_i + h_i^2 w_i \in K_0, \Big\}.$$

$$J^2(t) := \{(u,v) \mid \forall\, h_i \to 0+,\ \exists\,(u_i,v_i) \to (u,v), \\ \mathcal{H}_u[t]u_i = 0,\ \bar{u}(t) + h_i u_i + h_i^2 v_i \in U(t)\}.$$

Then $t \rightsquigarrow J^2(t)$ is measurable with closed images.

New Integral Necessary Conditions

Admissible variations of $\bar{u}(\cdot)$

$$\mathcal{M}^{(2)}(\bar{u}) := \{(u,v) \in L^2 \times L^1 \mid u \in \mathcal{C}_{loc}(\bar{u}), \\ (u(t), v(t)) \in J^2(t) \text{ a.e. in } A\}$$

Then $M^2(\bar{u}) \subset \mathcal{M}^{(2)}(\bar{u})$.

Theorem

If (SA), (ARA) hold true, then $\forall\, (u,v) \in \mathcal{M}^{(2)}(\bar{u})$, $\forall\, (y_0, w_0) \in J^2_{K_0}(\bar{x}(0))$

$$\Phi(u, v, y_0, w_0) \geq 0.$$