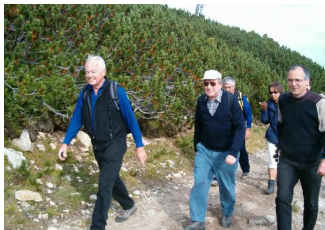




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Calmness as a constraint qualification for MPECs

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$$\min\{\varphi(x, y) \mid 0 \in F(x, y) + N_{\Gamma}(y)\} \quad (MPEC)$$

$\varphi : \mathbb{R}^{n+m} \rightarrow \mathbb{R}$, $F : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^m$ continuously differentiable, $\Gamma \subseteq \mathbb{R}^m$ closed, N normal cone

With $H(x, y) := (y, -F(x, y))$ equivalent with $\min\{\varphi(x, y) \mid (x, y) \in H^{-1}(\text{gr}N_{\Gamma})\}$.

Abstract M-stationarity conditions: $0 \in \nabla\varphi(\bar{x}, \bar{y}) + N_{H^{-1}(\text{gr}N_{\Gamma})}(\bar{x}, \bar{y})$.

Calmness of the mapping $M(p) := \{(x, y) \mid p \in F(x, y) + N_{\Gamma}(y)\}$ at $(0, \bar{x}, \bar{y})$ implies inclusion¹

$$N_{H^{-1}(\text{gr}N_{\Gamma})}(\bar{x}, \bar{y}) \subseteq (\nabla H(\bar{x}, \bar{y}))^T [N_{\text{gr}N_{\Gamma}}(\bar{y}, -F(\bar{x}, \bar{y}))]$$

Theorem (Ye/Ye 1997)

If (\bar{x}, \bar{y}) is a local solution to (*) and M is calm at $(0, \bar{x}, \bar{y})$,

then there are $(u^*, v^*) \in N_{\text{gr}N_{\Gamma}}(\bar{y}, -F(\bar{x}, \bar{y}))$ such that

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \nabla_x \varphi(\bar{x}, \bar{y}) \\ \nabla_y \varphi(\bar{x}, \bar{y}) \end{pmatrix} + \begin{pmatrix} -(\nabla_x F(\bar{x}, \bar{y}))^T v^* \\ u^* - (\nabla_y F(\bar{x}, \bar{y}))^T v^* \end{pmatrix}$$

¹H./Jourani/Outrata 2002

Definition

Let X, Y be metric spaces and $F : X \rightrightarrows Y$ a multifunction. Fix a point $(\bar{x}, \bar{y}) \in \text{gr}F$.

Then, F has the **Aubin property** at (\bar{x}, \bar{y}) , if there are $L \geq 0, \varepsilon > 0$ such that

$$d(y, F(x_2)) \leq Ld(x_1, x_2) \quad \forall y \in F(x_1) \cap \mathbb{B}_\varepsilon(\bar{y}) \quad \forall x_1, x_2 \in \mathbb{B}_\varepsilon(\bar{x})$$

Definition

Let X, Y be metric spaces and $F : X \rightrightarrows Y$ a multifunction. Fix a point $(\bar{x}, \bar{y}) \in \text{gr}F$.

Then, F is **calm** at (\bar{x}, \bar{y}) , if there are $L \geq 0, \varepsilon > 0$ such that

$$d(y, F(\bar{x})) \leq Ld(x, \bar{x}) \quad \forall y \in F(x) \cap \mathbb{B}_\varepsilon(\bar{y}) \quad \forall x \in \mathbb{B}_\varepsilon(\bar{x})$$

We consider sets defined by a finite number of \mathcal{C}^2 - inequalities:

$$\Gamma = \{y \in \mathbb{R}^m \mid g_j(y) \leq 0 \quad j = 1, \dots, q\}$$

Let Γ as above satisfy the Mangansarian-Fromovitz Constraint Qualification (MFCQ). Then, our MPEC

$$\min\{\varphi(x, y) \mid 0 \in F(x, y) + N_{\Gamma}(y)\} \quad (*)$$

can be rewritten as an (enhanced) MPEC

$$\min\{\varphi(x, y) \mid 0 \in (F(x, y) + [\nabla g(y)]^T \lambda, -g(y)) + N_{\mathbb{R}^m \times \mathbb{R}_+^q}(y, \lambda)\} \quad (**)$$

(\bar{x}, \bar{y}) is a solution of (*), if and only if for any

$$\bar{\lambda} \in \Lambda(\bar{x}, \bar{y}) := \{\lambda \geq 0 \mid F(\bar{x}, \bar{y}) + [\nabla g(\bar{y})]^T \lambda = 0, \lambda^T g(\bar{y}) = 0\}$$

$(\bar{x}, \bar{y}, \bar{\lambda})$ is a solution of (**).

We may reapply the previous theorem to the enhanced MPEC.

Corollary

Let $(\bar{x}, \bar{y}, \bar{\lambda})$ be a local solution to the enhanced MPEC

$$\min\{\varphi(x, y) \mid 0 \in (F(x, y) + [\nabla g(y)]^T \lambda, -g(y)) + N_{\mathbb{R}^m \times \mathbb{R}_+^q}(y, \lambda)\} \quad (**)$$

If

$$\tilde{M}(p_1, p_2) := \{(x, y, \lambda) \mid p_1 = F(x, y) + [\nabla g(y)]^T \lambda, p_2 \in g(y) + N_{\mathbb{R}_+^q}(\lambda)\}$$

is calm at $(0, 0, \bar{x}, \bar{y}, \bar{\lambda})$,

then there exists an MPEC multiplier $(v_1^*, v_2^*) \in \mathbb{R}^m \times \mathbb{R}^q$ such that

$$\begin{aligned} 0 &= \nabla_x \varphi(\bar{x}, \bar{y}) - [\nabla_x F(\bar{x}, \bar{y})]^T v_1^* \\ 0 &\in \nabla_y \varphi(\bar{x}, \bar{y}) - \left([\nabla_y F(\bar{x}, \bar{y})] + \sum_{i=1}^q \bar{\lambda}_i \nabla^2 g_i(\bar{y}) \right)^T v_1^* + [\nabla g(\bar{y})]^T v_2^* \\ &\quad (0, \nabla g(\bar{y})v_1^*, v_1^*, v_2^*) \in N_{\text{gr } N_{\mathbb{R}^m \times \mathbb{R}_+^q}}(\bar{y}, \bar{\lambda}, -F(\bar{x}, \bar{y}) - [\nabla g(\bar{y})]^T \bar{\lambda}, g(\bar{y})) \end{aligned}$$

The normal cone $N_{\text{gr } N_{\mathbb{R}^m \times \mathbb{R}_+^q}}$ has a simple explicit expression.

- How do the solutions compare? (answered)
- How do the stationarity conditions compare?

Stationarity conditions of enhanced MPEC are fully explicit

Stationarity conditions of initial MPEC are more precise

If Γ satisfies (LICQ), then both stationarity conditions coincide.

- How do the calmness assumptions compare?
- How to verify the calmness assumptions?

Comparison of calmness for canonical and enhanced perturbation mapping

For $\Gamma = \{y \in \mathbb{R}^m \mid g_j(y) \leq 0 \quad j = 1, \dots, q\}$ with $g \in \mathcal{C}^2$ and satisfying (MFCQ)

we derived M-stationarity conditions under two different calmness conditions:

$M(p) := \{(x, y) \mid p \in F(x, y) + N_\Gamma(y)\}$ is calm at $(0, \bar{x}, \bar{y})$ **canonical perturbation mapping**

$\tilde{M}(p_1, p_2) := \{(x, y, \lambda) \mid p_1 = F(x, y) + [\nabla g(y)]^T \lambda, p_2 \in g(y) + N_{\mathbb{R}_+^q}(\lambda)\}$ is calm at $(0, 0, \bar{x}, \bar{y}, \lambda)$ for some $\lambda \in \Lambda(\bar{x}, \bar{y}) := \{\lambda \geq 0 \mid F(\bar{x}, \bar{y}) + [\nabla g(\bar{y})]^T \lambda = 0, \lambda^T g(\bar{y}) = 0\}$ **enhanced perturbation mapping**

Theorem

1. Calmness of \tilde{M} at $(0, 0, \bar{x}, \bar{y}, \lambda)$ for all $\lambda \in \Lambda(\bar{x}, \bar{y})$ implies calmness of M at $(0, \bar{x}, \bar{y})$.
2. Let Γ be polyhedral or satisfy the **full rank constraint qualification** at \bar{y} :

$$\text{rank} \{ \nabla g_j(\bar{x}) \mid j \in I \} = \min \{ \#I, m \} \quad \forall I \subseteq I(\bar{y})$$

Then, M is calm at $(0, \bar{x}, \bar{y}) \iff \tilde{M}$ at $(0, 0, \bar{x}, \bar{y}, \lambda)$ is calm for all $\lambda \in \Lambda(\bar{x}, \bar{y})$.

- Converse of 1. is wrong even in the strong sense!
- 2. is wrong for $g \in \mathcal{C}^1$ even under (LICQ) for Γ

Example

Let

$$\Gamma = \{y \in \mathbb{R}^2 \mid g_1(y) \leq 0, g_2(y) \leq 0\},$$

where $g_j(y) := \varphi_j(y_1) - y_2$ ($j = 1, 2$) and

$$\varphi_1(t) := \begin{cases} (-1)^k \left(t - \frac{1}{k}\right)^3 \left(t - \frac{1}{k+1}\right)^3 & \text{for } t \in \left[\frac{1}{k+1}, \frac{1}{k}\right] \\ 0 & \text{else} \end{cases} \quad (k \in \mathbb{N})$$

$$\varphi_2(t) := \begin{cases} (-1)^k \left(t - \frac{1}{k}\right)^5 \left(t - \frac{1}{k+1}\right)^5 & \text{for } t \in \left[\frac{1}{k+1}, \frac{1}{k}\right] \\ 0 & \text{else} \end{cases} \quad (k \in \mathbb{N})$$

Moreover, let $F(x, y) := (-\varphi'(y_1), 1)$ for $\varphi(t) := \max\{\varphi_1(t), \varphi_2(t)\}$.

Then, $g_1, g_2 \in \mathcal{C}^2, F \in \mathcal{C}^1$ and Γ satisfies (MFCQ).

$M(p) := \{(x, y) \mid p \in F(x, y) + N_\Gamma(y)\}$ is calm at $(0, 0, 0)$.

$\tilde{M}(p_1, p_2) := \{(x, y, \lambda) \mid p_1 = F(x, y) + [\nabla g(y)]^T \lambda, p_2 \in g(y) + N_{\mathbb{R}_+^q}(\lambda)\}$

is **not calm at any** $(0, 0, 0, 0, \lambda)$ for $\lambda \in \Lambda(0, 0) = \{\lambda \geq 0 \mid \lambda_1 + \lambda_2 = 1\}$.

Theorem

The enhanced perturbation mapping

$$\tilde{M}(p_1, p_2) := \{(x, y, \lambda) | p_1 = F(x, y) + [\nabla g(y)]^T \lambda, p_2 \in g(y) + N_{\mathbb{R}_+^q}(\lambda)\}$$

is calm at $(0, 0, \bar{x}, \bar{y}, \lambda)$ for some $\lambda \in \Lambda(\bar{x}, \bar{y})$ under the following two conditions:

1. For all a, c the following implication holds true:

$$\left. \begin{aligned} 0 &= (\nabla_x F(\bar{x}, \bar{y}))^T a \\ 0 &= ([\nabla_y F(\bar{x}, \bar{y})] + \sum_{i=1}^q \lambda_i \nabla^2 g_i(\bar{y}))^T a + [\nabla g(\bar{y})]^T c \\ &(-\nabla g(\bar{y})a, c) \in N_{\text{gr } N_{\mathbb{R}_+^q}}(\lambda, g(\bar{y})) \end{aligned} \right\} \Rightarrow a = 0$$

2. For $I(\lambda) := \{i \in \{1, \dots, q\} | \lambda_i > 0\}$ the following mapping is calm at $(0, \bar{y})$:

$$T_{I(\lambda)}(p) := \{y | g_i(y) = p_i \quad (i \in I(\lambda)), \quad g_i(y) \leq p_i \quad (i \in I^c(\lambda))\}$$

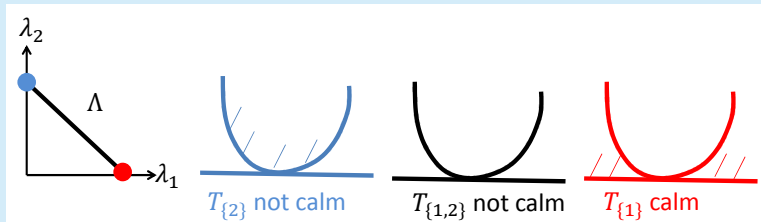
- Strengthening the implication in 1. to $' \Rightarrow a = 0, c = 0'$ yields GMFCQ, NNAMCQ.
- Under CRCQ for Γ already 2. is automatic and already 1. implies calmness of even M.

Example

In the generalized equation $0 \in F(x, y) + N_{\Gamma}(y)$ let

$$F(x, y_1, y_2) := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \Gamma := \{y | y_2 \geq y_1^2, y_2 \geq 0\}$$

One may easily show that the canonical perturbation mapping M is calm at $(0, 0, 0, 0, 0)$.



First condition (implication) of Theorem satisfied for $\lambda := (1, 0)$. Therefore, Theorem yields calmness of \tilde{M} at $(0, 0, 0, 0, 0, 1, 0)$.

\tilde{M} is not calm at $(0, 0, 0, 0, 0, 0, 1)$.

Although conditions of Theorem are not satisfied, \tilde{M} is calm at $(0, 0, 0, 0, 0, \lambda_1, \lambda_2)$ for $\lambda_1, \lambda_2 > 0$.

Calmness of $M(p) := \{(x, y) | p \in F(x, y) + N_{\Gamma}(y)\}$ is automatic under **polyhedrality** or under the stronger **Aubin property**.

- **Aubin property** via Mordukhovich criterion:

$$[\nabla_x F(\bar{x}, \bar{y})]^T v^* = 0, \left([\nabla_y F(\bar{x}, \bar{y})]^T v^*, v^* \right) \in N_{\text{gr } N_{\Gamma}(\bar{y}, -F(\bar{x}, \bar{y}))} \implies v^* = 0$$

- **Polyhedrality**: F affine linear and Γ polyhedron (consequence of Robinson's Theorem)
- **Structured calmness** (mix both cases)

Theorem (Klatte/Kummer 2002)

Let $T_1 : X_1 \rightrightarrows X$ and $T_2 : X_2 \rightrightarrows X$ be multifunctions between metric spaces X_1, X_2, X . If

1. T_1 is calm at $(x_1, x) \in \text{gr } T_1$
2. T_2 is calm at $(x_2, x) \in \text{gr } T_2$
3. T_2^{-1} has the Aubin property at (x, x_2)
4. $T_1(x_1) \cap T_2(\cdot)$ is calm at (x_2, x) ,

then the multifunction $(T_1 \cap T_2)(x_1, x_2) := T_1(x_1) \cap T_2(x_2)$ is calm at (x_1, x_2, x) .

Proposition

Consider the canonical perturbation mapping $M(p) := \{(x, y) | p \in F(x, y) + N_{\Gamma}(y)\}$.

Fix any $(\bar{x}, \bar{y}) \in M(0)$. Assume that Γ is polyhedral and

$$F(x, y) = \begin{pmatrix} F_1(x, y) \\ F_2(y) \end{pmatrix} \quad \text{with } F_2 \text{ affine linear and } \nabla_x F_1(\bar{x}, \bar{y}) \text{ surjective.}$$

Then, M is calm at $(0, \bar{x}, \bar{y})$.

Application to a class of bilevel problems

We consider the following class of bilevel problems (optimistic form):

$$\min\{\varphi(x, y) \mid y \in \operatorname{argmin}\{\underbrace{\langle x^a, y^a \rangle + \delta(x^b, y^a) + \langle y^b, Cy^b \rangle + \langle c, y^b \rangle}_{f(x, y)} \mid y \in \Gamma\}\} \quad (\star)$$

Here, $x = (x^a, x^b)$, $y = (y^a, y^b)$, C is an arbitrary positive semi-definite matrix (no rank assumption), δ is an arbitrary twice continuously differentiable function and Γ is polyhedral.

Applies to M-stationarity conditions for EPECs in electricity spot market models ².

Theorem

Let $(\bar{x}, \bar{y}^a, \bar{y}^b)$ be a solution to (\star) . Then (without any further constraint qualification):

Then, there exists some bilevel problem multipliers v^* , $u^* = (u_1^*, u_2^*)$ such that:

$$\begin{aligned} 0 &= \nabla_{y^a} \varphi(\bar{x}, \bar{y}) - \nabla_{y^a y^a}^2 \delta(\bar{x}^b, \bar{y}^a) \nabla_x \varphi(\bar{x}, \bar{y}) + u_1^* \\ 0 &= \nabla_{y^b} \varphi(\bar{x}, \bar{y}) + (C + C^T) v^* + u_2^* \\ 0 &= \nabla_{x^b} \varphi(\bar{x}, \bar{y}) - \nabla_{y^a x^b}^2 \delta(\bar{x}^b, \bar{y}^a) \nabla_x \varphi(\bar{x}, \bar{y}) \\ &\quad (u^*, \nabla_x \varphi(\bar{x}, \bar{y}), v^*) \in N_{\operatorname{gr} N_\Gamma(\bar{y}, -F(\bar{x}, \bar{y}))} \end{aligned}$$

Red part is easily made explicit due to Γ being polyhedral (Dontchev/Rockafellar '96)

²H./Oustrata/Surowiec 2012