Weierstrass Institute for
Applied Analysis and Stochastics

Calmness as a constraint qualification for MPECs

Terryfest 2015, Limoges

R. Henrion
Weierstrass Institut Berlin
Joint work with L. Adam and J. Outrata (UTIA Prague)

$$\min\{\varphi(x,y) | 0 \in F(x,y) + N_\Gamma(y)\} \qquad (MPEC)$$

$\varphi : \mathbb{R}^{n+m} \to \mathbb{R}$, $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ continuously differentiable, $\Gamma \subseteq \mathbb{R}^m$ closed, N normal cone

With $H(x,y) := (y, -F(x,y))$ equivalent with $\min\{\varphi(x,y) | (x,y) \in H^{-1}(\mathrm{gr} N_\Gamma)\}$.

Abstract M-stationarity conditions: $0 \in \nabla\varphi(\bar{x},\bar{y}) + N_{H^{-1}(\mathrm{gr} N_\Gamma)}(\bar{x},\bar{y})$.

Calmness of the mapping $M(p) := \{(x,y) | p \in F(x,y) + N_\Gamma(y)\}$ at $(0,\bar{x},\bar{y})$ implies inclusion[1]

$$N_{H^{-1}(\mathrm{gr} N_\Gamma)}(\bar{x},\bar{y}) \subseteq (\nabla H(\bar{x},\bar{y}))^T \left[N_{\mathrm{gr} N_\Gamma}(\bar{y}, -F(\bar{x},\bar{y}))\right]$$

Theorem (Ye/Ye 1997)

If $(\bar{x},\bar{y})$ is a local solution to (*) and M is calm at $(0,\bar{x},\bar{y})$,

then there are $(u^*, v^*) \in N_{\mathrm{gr} N_\Gamma}(\bar{y}, -F(\bar{x},\bar{y}))$ such that

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \nabla_x \varphi(\bar{x},\bar{y}) \\ \nabla_y \varphi(\bar{x},\bar{y}) \end{pmatrix} + \begin{pmatrix} -(\nabla_x F(\bar{x},\bar{y}))^T v^* \\ u^* - (\nabla_y F(\bar{x},\bar{y}))^T v^* \end{pmatrix}$$

[1] H./Jourani/Outrata 2002

Definition

Let X, Y be metric spaces and $F : X \rightrightarrows Y$ a multifunction. Fix a point $(\bar{x}, \bar{y}) \in \mathrm{grF}$.

Then, F has the **Aubin property** at $(\bar{x}, \bar{y})$, if there are $L \geq 0, \varepsilon > 0$ such that

$$d(y, F(x_2)) \leq Ld(x_1, x_2) \quad \forall y \in F(x_1) \cap \mathbb{B}_\varepsilon(\bar{y}) \; \forall x_1, x_2 \in \mathbb{B}_\varepsilon(\bar{x})$$

Definition

Let X, Y be metric spaces and $F : X \rightrightarrows Y$ a multifunction. Fix a point $(\bar{x}, \bar{y}) \in \mathrm{grF}$.

Then, F is **calm** at $(\bar{x}, \bar{y})$, if there are $L \geq 0, \varepsilon > 0$ such that

$$d(y, F(\bar{x})) \leq Ld(x, \bar{x}) \quad \forall y \in F(x) \cap \mathbb{B}_\varepsilon(\bar{y}) \; \forall x \in \mathbb{B}_\varepsilon(\bar{x})$$

Enhanced MPEC

We consider sets defined by a finite number of $\mathcal{C}^2$- inequalities:

$$\Gamma = \{y \in \mathbb{R}^m | g_j(y) \leq 0 \quad j = 1, \ldots, q\}$$

Let Γ as above satisfy the Mangansarian-Fromovitz Constraint Qualification (MFCQ). Then, our MPEC

$$\min\{\varphi(x,y) | 0 \in F(x,y) + N_\Gamma(y)\} \quad (*)$$

can be rewritten as an (enhanced) MPEC

$$\min\{\varphi(x,y) | 0 \in (F(x,y) + [\nabla g(y)]^T \lambda, -g(y)) + N_{\mathbb{R}^m \times \mathbb{R}^q_+}(y,\lambda)\} \quad (**)$$

$(\bar{x}, \bar{y})$ is a solution of (*), if and only if for any

$$\bar{\lambda} \in \Lambda(\bar{x}, \bar{y}) := \{\lambda \geq 0 \mid F(\bar{x}, \bar{y}) + [\nabla g(\bar{y})]^T \lambda = 0, \ \lambda^T g(\bar{y}) = 0\}$$

$(\bar{x}, \bar{y}, \bar{\lambda})$ is a solution of (**).

We may reapply the previous theorem to the enhanced MPEC.

M-stationarity conditions for the enhanced MPEC

Corollary

Let $(\bar{x}, \bar{y}, \bar{\lambda})$ *be a local solution to the enhanced MPEC*

$$\min\{\varphi(x,y) | 0 \in (F(x,y) + [\nabla g(y)]^T \lambda, -g(y)) + N_{\mathbb{R}^m \times \mathbb{R}^q_+}(y, \lambda)\} \quad (**)$$

If

$$\tilde{M}(p_1, p_2) := \{(x,y,\lambda) | p_1 = F(x,y) + [\nabla g(y)]^T \lambda,\ p_2 \in g(y) + N_{\mathbb{R}^q_+}(\lambda)\}$$

is calm at $(0, 0, \bar{x}, \bar{y}, \bar{\lambda})$,

then there exists an MPEC multiplier $(v_1^*, v_2^*) \in \mathbb{R}^m \times \mathbb{R}^q$ *such that*

$$\begin{aligned} 0 &= \nabla_x \varphi(\bar{x}, \bar{y}) - [\nabla_x F(\bar{x}, \bar{y})]^T v_1^* \\ 0 &\in \nabla_y \varphi(\bar{x}, \bar{y}) - \left([\nabla_y F(\bar{x}, \bar{y})] + \sum_{i=1}^q \bar{\lambda}_i \nabla^2 g_i(\bar{y}) \right)^T v_1^* + [\nabla g(\bar{y})]^T v_2^* \\ & (0, \nabla g(\bar{y}) v_1^*, v_1^*, v_2^*) \in N_{\mathrm{gr}\, N_{\mathbb{R}^m \times \mathbb{R}^q_+}}(\bar{y}, \bar{\lambda}, -F(\bar{x}, \bar{y}) - [\nabla g(\bar{y})]^T \bar{\lambda}, g(\bar{y})) \end{aligned}$$

The normal cone $N_{\mathrm{gr}\, N_{\mathbb{R}^m \times \mathbb{R}^q_+}}$ has a simple explicit expression.

Arising questions when comparing the initial and enhanced MPECs

- How do the solutions compare? (answered)
- How do the stationarity conditions compare?

  Stationarity conditions of enhanced MPEC are fully explicit
  Stationarity conditions of initial MPEC are more precise
  If Γ satisfies (LICQ), then both stationarity conditions coincide.

- How do the calmness assumptions compare?
- How to verify the calmness assumptions?

Comparison of calmness for canonical and enhanced perturbation mapping

For $\Gamma = \{y \in \mathbb{R}^m | g_j(y) \leq 0 \quad j = 1, \dots, q\}$ with $g \in \mathcal{C}^2$ and satisfying (MFCQ)

we derived M-stationarity conditions under two different calmness conditions:

$M(p) := \{(x, y) | p \in F(x, y) + N_\Gamma(y)\}$ is calm at $(0, \bar{x}, \bar{y})$ canonical perturbation mapping

$\tilde{M}(p_1, p_2) := \{(x, y, \lambda) | p_1 = F(x, y) + [\nabla g(y)]^T \lambda,\ p_2 \in g(y) + N_{\mathbb{R}^q_+}(\lambda)\}$ is calm at $(0, 0, \bar{x}, \bar{y}, \lambda)$ for some $\lambda \in \Lambda(\bar{x}, \bar{y}) := \{\lambda \geq 0 \mid F(\bar{x}, \bar{y}) + [\nabla g(\bar{y})]^T \lambda = 0,\ \lambda^T g(\bar{y}) = 0\}$
enhanced perturbation mapping

Theorem

1. *Calmness of* $\tilde{M}$ *at* $(0, 0, \bar{x}, \bar{y}, \lambda)$ *for* **all** $\lambda \in \Lambda(\bar{x}, \bar{y})$ *implies calmness of* M *at* $(0, \bar{x}, \bar{y})$.
2. *Let* Γ *be polyhedral or satisfy the* **full rank constraint qualification** *at* $\bar{y}$:

$$\operatorname{rank}\{\nabla g_j(\bar{x}) | j \in I\} = \min\{\#I, m\} \quad \forall I \subseteq I(\bar{y})$$

Then, M *is calm at* $(0, \bar{x}, \bar{y}) \Longleftrightarrow \tilde{M}$ *at* $(0, 0, \bar{x}, \bar{y}, \lambda)$ *is calm for* **all** $\lambda \in \Lambda(\bar{x}, \bar{y})$.

- Converse of 1. is wrong even in the strong sense!
- 2. is wrong for $g \in \mathcal{C}^1$ even under (LICQ) for Γ

Example

Let

$$\Gamma = \{y \in \mathbb{R}^2 | g_1(y) \leq 0,\ g_2(y) \leq 0\},$$

where $g_j(y) := \varphi_j(y_1) - y_2 \quad (j = 1, 2)$ and

$$\varphi_1(t) := \begin{cases} (-1)^k \left(t - \frac{1}{k}\right)^3 \left(t - \frac{1}{k+1}\right)^3 & \text{for } t \in \left[\frac{1}{k+1}, \frac{1}{k}\right] \\ 0 & \text{else} \end{cases} \qquad (k \in \mathbb{N})$$

$$\varphi_2(t) := \begin{cases} (-1)^k \left(t - \frac{1}{k}\right)^5 \left(t - \frac{1}{k+1}\right)^5 & \text{for } t \in \left[\frac{1}{k+1}, \frac{1}{k}\right] \\ 0 & \text{else} \end{cases} \qquad (k \in \mathbb{N})$$

Moreover, let $F(x, y) := (-\varphi'(y_1), 1)$ for $\varphi(t) := \max\{\varphi_1(t), \varphi_2(t)\}$.

Then, $g_1, g_2 \in \mathcal{C}^2, F \in \mathcal{C}^1$ and Γ satisfies (MFCQ).

$M(p) := \{(x, y) | p \in F(x, y) + N_\Gamma(y)\}$ is calm at $(0, 0, 0)$.

$\tilde{M}(p_1, p_2) := \{(x, y, \lambda) | p_1 = F(x, y) + [\nabla g(y)]^T \lambda,\ p_2 \in g(y) + N_{\mathbb{R}^q_+}(\lambda)\}$

is **not calm at any** $(0, 0, 0, 0, \lambda)$ for $\lambda \in \Lambda(0, 0) = \{\lambda \geq 0 \mid \lambda_1 + \lambda_2 = 1\}$.

Theorem

The enhanced perturbation mapping

$$\tilde{M}(p_1, p_2) := \{(x, y, \lambda) | p_1 = F(x, y) + [\nabla g(y)]^T \lambda,\ p_2 \in g(y) + N_{\mathbb{R}^q_+}(\lambda)\}$$

is calm at $(0, 0, \bar{x}, \bar{y}, \lambda)$ *for some* $\lambda \in \Lambda(\bar{x}, \bar{y})$ *under the following two conditions:*

1. *For all* a, c *the following implication holds true:*

$$\left.\begin{array}{rcl} 0 & = & (\nabla_x F(\bar{x}, \bar{y}))^T a \\ 0 & = & \left([\nabla_y F(\bar{x}, \bar{y})] + \sum_{i=1}^q \lambda_i \nabla^2 g_i(\bar{y})\right)^T a + [\nabla g(\bar{y})]^T c \\ & & (-\nabla g(\bar{y})a, c) \in N_{\mathrm{gr}\, N_{\mathbb{R}^q_+}}(\lambda, g(\bar{y})) \end{array}\right\} \Rightarrow a = 0$$

2. *For* $I(\lambda) := \{i \in \{1, \ldots, q\} | \lambda_i > 0\}$ *the following mapping is calm at* $(0, \bar{y})$*:*

$$T_{I(\lambda)}(p) := \{y | g_i(y) = p_i \quad (i \in I(\lambda)), \quad g_i(y) \le p_i \quad (i \in I^c(\lambda))\}$$

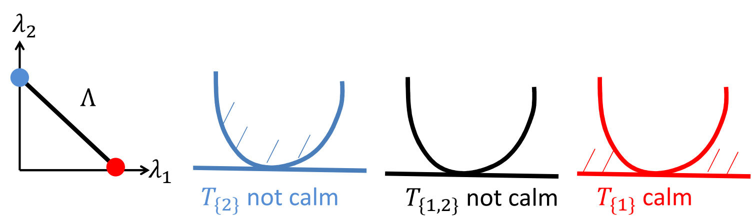
- Strengthening the implication in 1. to '$\Rightarrow a = 0, c = 0$' yields GMFCQ, NNAMCQ.
- Under CRCQ for Γ already 2. is automatic and already 1. implies calmness of even M.

Example

In the generalized equation $0 \in F(x, y) + N_\Gamma(y)$ let

$$F(x, y_1, y_2) := \binom{0}{1}, \quad \Gamma := \{y | y_2 \geq y_1^2 \; y_2 \geq 0\}$$

One may easily show that the canonical perturbation mapping M is calm at $(0, 0, 0, 0, 0)$.



First condition (implication) of Theorem satisfied for $\lambda := (1, 0)$. Therefore, Theorem yields calmness of $\tilde{M}$ at $(0, 0, 0, 0, 0, 1, 0)$.
$\tilde{M}$ is not calm at $(0, 0, 0, 0, 0, 0, 1)$.
Although conditions of Theorem are not satisfied, $\tilde{M}$ is calm at $(0, 0, 0, 0, 0, \lambda_1, \lambda_2)$ for $\lambda_1, \lambda_2 > 0$.

Verifiable conditions for calmness of the canonical perturbation mapping

Calmness of $M(p) := \{(x,y)|p \in F(x,y) + N_\Gamma(y)\}$ is automatic under polyhedrality or under the stronger Aubin property.

- Aubin property via Mordukhovich criterion:

$$[\nabla_x F(\bar{x},\bar{y})]^T v^* = 0, \ \left([\nabla_y F(\bar{x},\bar{y})]^T v^*, v^*\right) \in N_{\mathrm{gr}\, N_\Gamma(\bar{y}, -F(\bar{x},\bar{y})} \Longrightarrow v^* = 0$$

- Polyhedrality: F affine linear and Γ polyhedron (consequence of Robinson's Theorem)
- Structured calmness (mix both cases)

Theorem (Klatte/Kummer 2002)

Let $T_1 : X_1 \rightrightarrows X$ and $T_2 : X_2 \rightrightarrows X$ be multifunctions between metric spaces X_1, X_2, X. If

1. T_1 *is calm at* $(x_1, x) \in \mathrm{gr}\, T_1$
2. T_2 *is calm at* $(x_2, x) \in \mathrm{gr}\, T_2$
3. T_2^{-1} *has the Aubin property at* (x, x_2)
4. $T_1(x_1) \cap T_2(\cdot)$ *is calm at* (x_2, x),

then the multifunction $(T_1 \cap T_2)(x_1, x_2) := T_1(x_1) \cap T_2(x_2)$ *is calm at* (x_1, x_2, x).

Structured calmness

Proposition

Consider the canonical perturbation mapping $M(p) := \{(x,y) | p \in F(x,y) + N_\Gamma(y)\}$.

Fix any $(\bar{x},\bar{y}) \in M(0)$. *Assume that* Γ *is polyhedral and*

$$F(x,y) = \begin{pmatrix} F_1(x,y) \\ F_2(y) \end{pmatrix} \quad \text{with } F_2 \text{ affine linear and } \nabla_x F_1(\bar{x},\bar{y}) \text{ surjective.}$$

Then, M *is calm at* $(0,\bar{x},\bar{y})$.

We consider the following class of bilevel problems (optimistic form):

$$\min\{\varphi(x,y)|y \in \operatorname{argmin}\{\underbrace{\langle x^a, y^a\rangle + \delta(x^b, y^a) + \langle y^b, Cy^b\rangle + \langle c, y^b\rangle}_{f(x,y)} |y \in \Gamma\}\} \qquad (\bigstar)$$

Here, $x = (x^a, x^b)$, $y = (y^a, y^b)$, C is an arbitrary positive semi-definite matrix (no rank assumption), δ is an arbitrary twice continuously differentiable function and Γ is polyhedral.

Applies to M-stationarity conditions for EPECs in electricity spot market models [2].

Theorem

Let $(\bar{x}, \bar{y^a}, \bar{y^b})$ be a solution to $(\bigstar)$. Then (without any further constraint qualification):

Then, there exists some bilevel problem multipliers v^*, $u^* = (u_1^*, u_2^*)$ such that:

$$\begin{aligned}
0 &= \nabla_{y^a}\varphi(\bar{x},\bar{y}) - \nabla^2_{y^a y^a}\delta(\bar{x}^b,\bar{y}^a)\nabla_x\varphi(\bar{x},\bar{y}) + u_1^* \\
0 &= \nabla_{y^b}\varphi(\bar{x},\bar{y}) + (C + C^T)v^* + u_2^* \\
0 &= \nabla_{x^b}\varphi(\bar{x},\bar{y}) - \nabla^2_{y^a x^b}\delta(\bar{x}^b,\bar{y}^a)\nabla_x\varphi(\bar{x},\bar{y}) \\
& \quad (u^*, \nabla_x\varphi(\bar{x},\bar{y}), v^*) \in N_{\operatorname{gr} N_\Gamma(\bar{y}, -F(\bar{x},\bar{y}))}
\end{aligned}$$

Red part is easily made explicit due to Γ being polyhedral (Dontchev/Rockafellar '96)

[2] H./Outrata/Surowiec 2012