Quadratic growth and subregularity of subdifferentials

(based on the joint work with D. Drusvyatskiy)

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Regularity and subregularity

$F : X \rightrightarrows Y$ a set-valued mappings; $\quad \overline{y} \in F(\overline{x})$

- F is *(metrically) regular near* $(\bar{x}, \bar{y})$ if

$$d(x, F^{-1}(y)) \leq Kd(y, F(x))$$

for all (x, y) of a neighborhood $(\bar{x}, \bar{y})$;

- F is *strongly (metrically) regular near* $(\bar{x}, \bar{y})$ if it is regular there and there is an $\varepsilon > 0$ s.t. $F(x) \cap F(x') \cap B(\overline{y}, \varepsilon) = \emptyset$ for all $x \neq x'$ near $\overline{x}$ (equivalently: $d(x, u) \leq Kd(y, F(x))$ if $y \in F(u)$);

- F is *(metrically) subregular near* $(\bar{x}, \bar{y})$ if

$$d(x, F^{-1}(\overline{y})) \leq Kd(\overline{y}, F(x))$$

for all x of a neighborhood $\overline{x}$;

- F is *strongly (metrically) subregular at* $(\bar{x}, \bar{y})$ if it is subregular near $(\bar{x}, \bar{y})$ and there is an $\varepsilon > 0$ s.t. $d(x, \overline{x}) \leq Kd(\overline{y}, F(x))$ for x of a neighborhood of $\overline{x}$.

The lower bound of K in each case is the *rate (modulus)* of the corresponding phenomenon.

Quadratic growth

Given a Banach space X, a function f on$X$ and a point $\overline{x}$ at which F is finite.
It is said that f has a *strong local minimum at* $\overline{x}$ if there is an $\alpha > 0$ such that

$$f(x) \geq f(\overline{x}) + \alpha \|x - \overline{x}\|^2$$

for all x of a neighborhood of $\overline{x}$.

Convex functions

Klatte & Kummer 2002 (Kluwer, monograph, Theorem 5.4): *if f is a convex function on $\mathbf{R}^n$ and $0 \in \partial f(\overline{x})$, then strong regularity of ∂f at $(\overline{x}, 0)$ is equivalent to the existence of an $\alpha > 0$ such that for all x of a neighborhood of $\overline{x}$*

$$f(x) \geq f(\overline{x}) + \frac{\alpha}{2}\|x - \overline{x}\|^2. \qquad (*)$$

Aragon Artacho & Geoffroy 2013 (J. Nonlinear and Convex Analysis): *X is a Banach space and f is a proper convex lsc function on X such that $0 \in \partial f(\overline{x})$, then there is an $\alpha > 0$ s.t.*

$$f(x) \geq f(\overline{x}) + \alpha d(x, (\partial f)^{-1}(0))^2$$

iff ∂f is subregular of at $(\overline{x}, 0)$.

Aragon-Artacho & Geoffroy (2013) and Drusvyatzkiy,Mordukhovich & Nghia 2014 (J. Convex Analysis) extended a part of the last result to lsc proper functions on Asplund spaces and the limiting Fréchet subdifferential.

Three results to be discussed

1. Characterization of strong minimality (Klatte-Kummer type) for semi-algebraic functions on $\mathbf{R}^n$;

2. Extension of Aragon-Geoffroy and Drusvyatskiy-Mordukhovich-Nghia results to arbitrary Banach spaces and arbitrary trusted subdifferentials;

3. Second order optimality conditions based on the first order subdifferentials.

Semi-algebraic functions

Recall: a set $Q \subset \mathbf{R}^n$ is semi-algebraic if it is a union of solutions of finitely many finite systems of polynomial equalities and inequalities of n variables (the same for each of the systems). A function, mapping is semi-algebraic if its graph is semi-algebraic.

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Theorem 1. *Let f be a lsc semi-algebraic function on $\mathbf{R}^n$, and let $\overline{x}$ be a local minimizer of f. Consider the following two statements:*
1. ∂f (the limiting subdifferential) is strongly subregular at $(\overline{x}, 0)$ with rate κ;
2. there are $\varepsilon > 0$, $\alpha > 0$ such that $(*)$ holds if $\|x - \overline{x}\| < \varepsilon$.

Then $1 \;\Rightarrow\; 2$ with any $\alpha \in (0, \kappa^{-1})$ and $2 \;\Rightarrow\; 1$ if f is subdifferentially continuous at $\overline{x}$ for 0.

Metric subregularity always entails quadratic growth

X is a Banach space, and ∂ is a subdifferential which is *trusted on* X, that is:

for any proper lsc function f on X, any $\overline{x} \in \operatorname{dom} f$ and any function g which is convex continuous near $\overline{x}$ the following holds: if $f + g$ has a local minimum at $\overline{x}$, then for any $\varepsilon > 0$ there are x, u ε-close to $\overline{x}$ and $x^* \in \partial f(x)$, $u^* \in \partial g(u)$ such that $\|x^* + u^*\| < \varepsilon$.

– proximal subdifferential is trusted on Hilbert spaces;
– Fréchet and limiting Fréchet subdifferentials are trusted on Asplund spaces and only on them;
– Dini-Hadamard subdifferential is trusted on Gâteaux smooth spaces;
– G-subdifferential and generalized gradient are trusted on all Banach spaces.
We say that ∂ has *exact calculus* if $\partial(f + g)(x) \subset \partial f(x) + \partial g(x)$ whenever f is lsc and g Lipschitz continuous at x.

Metric subregularity always entails quadratic growth

Theorem 2. *Consider a proper lsc function f on a Banach space X that has a local minimum at a certain $\overline{x} \in \operatorname{dom} f$. Let ∂ be a subdifferential trusted on X and suppose that ∂f is subregular at $\overline{x}$ with modulus κ. Then for any $\alpha \in (0, 1/2\kappa)$ there is an $\varepsilon > 0$ such that for any $x \in B(\overline{x}, \varepsilon)$*

$$f(x) \geq f(\overline{x}) + \frac{\alpha}{2} d(x, (\partial f)^{-1}(0))$$

Moreover, if ∂ has exact calculus, then the conclusion is valid for any $\alpha \in (0, \kappa^{-1})$.

A perturbation lemma

Given: Banach spaces X and Y and set-valued mappings $F, G : X \rightrightarrows Y$ with closed graphs. We assume that F is subregular at $(\bar{x}, \bar{y}) \in \mathrm{Graph}\ F$ with modulus κ and

$$G(x) \subset \ell d(x, F^{-1}(\overline{y}))B_Y$$

for all x of a neighborhood of $\overline{x}$. Then $F + G$ is subregular at $(\bar{x}, \bar{y})$ with modulus not greater than $(\kappa^{-1} - \ell)^{-1}$. Specifically, for all x of a neighborhood of $\overline{x}$

$$d(x, (F + G)^{-1}(\overline{y})) \leq (\kappa^{-1} - \ell)^{-1} d(\overline{y}, (F + G)(x)).$$

Non-standard view of standard 2d order conditions

Question: can we get a reasonable second order conditions from these quadratic growth characterizations?

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Let f be a C^2 function on $\mathbf{R}^n$ and $\nabla f(\overline{x}) = 0$. Then
1. If f has a local minimum at $\overline{x}$ then the mapping $x \mapsto \nabla f + r(x - \overline{x})$ is strongly subregular at $\overline{x}$ for any $r > 0$;
2. If there are $r > 0$ and $\lambda > 0$ such that

$$\|\nabla f(x) + r(x - \overline{x})\| \geq (\lambda + r)\|x - \overline{x}\|$$

for all x of a neighborhood of $\overline{x}$, then $\overline{x}$ is a strong local minimum of f.

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Actually, 1 is equivalent to $\nabla^2 f(\overline{x})$ being positive semidefinite and 2 – to f being positive definite.

Subdifferential characterization for a 2d order minorant

Worrying example: $f(x) = |x|^{\frac{3}{2}}$ – zero is a critical point and

$$|\frac{3}{2}\sqrt{|x|}\text{sign}x + rx| \geq (\lambda + r)|x|$$

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Solution: Set for a closed $K \subset \mathbf{R}^n$

$$\Gamma_K(x) = \inf\{\lambda \geq 0 : \ x \in \lambda K\}$$

(the a *gauge* or the *Minkowski function of* K) and

$$\Gamma_{\partial f}(x, y) = \Gamma_{\partial f(x)}(y).$$

Then *the condition* $\Gamma_{\hat{\partial} f}(x, x - \overline{x}) \geq \alpha > 0$ *is sufficient and, if* f *is semi-algebraic, the condition* $\Gamma_{\partial f}(x, x - \overline{x}) \geq \alpha$ *is necessary for* f *to have a quadratic minorant at* $\overline{x}$.

"First order" 2d order conditions

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Theorem 3. *Let f be a lsc function on $\mathbf{R}^n$ finite at $\overline{x}$ and $0 \in \partial f(\overline{x})$.*
1. If there are $\lambda > 0$ and $r > 0$ such that

$$d(0, \partial f(x) + r(x - \overline{x})) \geq (\lambda + r)\|x - \overline{x}\| \quad \& \quad \Gamma_{\hat{\partial} f}(x, x - \overline{x}) \geq \frac{1}{2r} > 0,$$

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then $\overline{x}$ is a strong local minimizer of f;

2. If in addition f is semi-algebraic, then a necessary condition for f to have a local minimum at $\overline{x}$ is the for every $r > 0$ the following properties hold:
(a) the set-valued mapping $x \mapsto \partial f(x) + r(x - \overline{x})$ is strongly subregular at $\overline{x}$;
(b) $\Gamma_{\partial f}(x, x - \overline{x}) \geq r^{-1}$ for all x of a neighborhood of $\overline{x}$.

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THANK YOU

AND VERY BEST WISHES TO TERRY