

Quadratic growth and subregularity of subdifferentials

(based on the joint work with D. Drusvyatskiy)

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Regularity and subregularity

$F : X \rightrightarrows Y$ a set-valued mappings; $\bar{y} \in F(\bar{x})$

- F is (metrically) regular near (\bar{x}, \bar{y}) if

$$d(x, F^{-1}(y)) \leq Kd(y, F(x))$$

for all (x, y) of a neighborhood (\bar{x}, \bar{y}) ;

- F is strongly (metrically) regular near (\bar{x}, \bar{y}) if it is regular there and there is an $\varepsilon > 0$ s.t. $F(x) \cap F(x') \cap B(\bar{y}, \varepsilon) = \emptyset$ for all $x \neq x'$ near \bar{x} (equivalently: $d(x, u) \leq Kd(y, F(x))$ if $y \in F(u)$);

- F is (metrically) subregular near (\bar{x}, \bar{y}) if

$$d(x, F^{-1}(\bar{y})) \leq Kd(\bar{y}, F(x))$$

for all x of a neighborhood \bar{x} ;

- F is strongly (metrically) subregular at (\bar{x}, \bar{y}) if it is subregular near (\bar{x}, \bar{y}) and there is an $\varepsilon > 0$ s.t.

$d(x, \bar{x}) \leq Kd(\bar{y}, F(x))$ for x of a neighborhood of \bar{x} .

The lower bound of K in each case is the *rate (modulus)* of the corresponding phenomenon.

Quadratic growth

Given a Banach space X , a function f on X and a point \bar{x} at which F is finite.

It is said that f has a *strong local minimum at \bar{x}* if there is an $\alpha > 0$ such that

$$f(x) \geq f(\bar{x}) + \alpha \|x - \bar{x}\|^2$$

for all x of a neighborhood of \bar{x} .

Convex functions

Klatte & Kummer 2002 (Kluwer, monograph, Theorem 5.4): *if f is a convex function on \mathbf{R}^n and $0 \in \partial f(\bar{x})$, then strong regularity of ∂f at $(\bar{x}, 0)$ is equivalent to the existence of an $\alpha > 0$ such that for all x of a neighborhood of \bar{x}*

$$f(x) \geq f(\bar{x}) + \frac{\alpha}{2} \|x - \bar{x}\|^2. \quad (*)$$

Aragon Artacho & Geoffroy 2013 (J. Nonlinear and Convex Analysis): *X is a Banach space and f is a proper convex lsc function on X such that $0 \in \partial f(\bar{x})$, then there is an $\alpha > 0$ s.t.*

$$f(x) \geq f(\bar{x}) + \alpha d(x, (\partial f)^{-1}(0))^2$$

iff ∂f is subregular of at $(\bar{x}, 0)$.

Aragon-Artacho & Geoffroy (2013) and Drusvyatzkiy, Mordukhovich & Nghia 2014 (J. Convex Analysis) extended a part of the last result to lsc proper functions on Asplund spaces and the limiting Fréchet subdifferential.

Three results to be discussed

1. Characterization of strong minimality (Klatte-Kummer type) for semi-algebraic functions on \mathbf{R}^n ;
2. Extension of Aragon-Geoffroy and Drusvyatskiy-Mordukhovich-Nghia results to arbitrary Banach spaces and arbitrary trusted subdifferentials;
3. Second order optimality conditions based on the first order subdifferentials.

Semi-algebraic functions

Recall: a set $Q \subset \mathbf{R}^n$ is **semi-algebraic** if it is a union of solutions of finitely many finite systems of polynomial equalities and inequalities of n variables (the same for each of the systems). A function, mapping is semi-algebraic if its graph is semi-algebraic.

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Theorem 1. *Let f be a lsc semi-algebraic function on \mathbf{R}^n , and let \bar{x} be a local minimizer of f . Consider the following two statements:*

1. ∂f (the limiting subdifferential) is strongly subregular at $(\bar{x}, 0)$ with rate κ ;
2. there are $\varepsilon > 0$, $\alpha > 0$ such that (*) holds if $\|x - \bar{x}\| < \varepsilon$.

Then $1 \Rightarrow 2$ with any $\alpha \in (0, \kappa^{-1})$ and $2 \Rightarrow 1$ if f is subdifferentially continuous at \bar{x} for 0.

Metric subregularity always entails quadratic growth

X is a Banach space, and ∂ is a subdifferential which is *trusted* on X , that is:

for any proper lsc function f on X , any $\bar{x} \in \text{dom } f$ and any function g which is convex continuous near \bar{x} the following holds: if $f + g$ has a local minimum at \bar{x} , then for any $\varepsilon > 0$ there are x, u ε -close to \bar{x} and $x^* \in \partial f(x)$, $u^* \in \partial g(u)$ such that $\|x^* + u^*\| < \varepsilon$.

- proximal subdifferential is trusted on Hilbert spaces;
- Fréchet and limiting Fréchet subdifferentials are trusted on Asplund spaces and only on them;
- Dini-Hadamard subdifferential is trusted on Gâteaux smooth spaces;
- G -subdifferential and generalized gradient are trusted on all Banach spaces.

We say that ∂ has *exact calculus* if $\partial(f + g)(x) \subset \partial f(x) + \partial g(x)$ whenever f is lsc and g Lipschitz continuous at x .

Metric subregularity always entails quadratic growth

Theorem 2. *Consider a proper lsc function f on a Banach space X that has a local minimum at a certain $\bar{x} \in \text{dom } f$. Let ∂ be a subdifferential trusted on X and suppose that ∂f is subregular at \bar{x} with modulus κ . Then for any $\alpha \in (0, 1/2\kappa)$ there is an $\varepsilon > 0$ such that for any $x \in B(\bar{x}, \varepsilon)$*

$$f(x) \geq f(\bar{x}) + \frac{\alpha}{2} d(x, (\partial f)^{-1}(0))$$

Moreover, if ∂ has exact calculus, then the conclusion is valid for any $\alpha \in (0, \kappa^{-1})$.

A perturbation lemma

Given: Banach spaces X and Y and set-valued mappings $F, G : X \rightrightarrows Y$ with closed graphs. We assume that F is subregular at $(\bar{x}, \bar{y}) \in \text{Graph } F$ with modulus κ and

$$G(x) \subset \ell d(x, F^{-1}(\bar{y}))B_Y$$

for all x of a neighborhood of \bar{x} . Then $F + G$ is subregular at (\bar{x}, \bar{y}) with modulus not greater than $(\kappa^{-1} - \ell)^{-1}$. Specifically, for all x of a neighborhood of \bar{x}

$$d(x, (F + G)^{-1}(\bar{y})) \leq (\kappa^{-1} - \ell)^{-1} d(\bar{y}, (F + G)(x)).$$

Non-standard view of standard 2d order conditions

Question: can we get a reasonable second order conditions from these quadratic growth characterizations?

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Let f be a C^2 function on \mathbf{R}^n and $\nabla f(\bar{x}) = 0$. Then

1. If f has a local minimum at \bar{x} then the mapping $x \mapsto \nabla f + r(x - \bar{x})$ is strongly subregular at \bar{x} for any $r > 0$;
2. If there are $r > 0$ and $\lambda > 0$ such that

$$\|\nabla f(x) + r(x - \bar{x})\| \geq (\lambda + r)\|x - \bar{x}\|$$

for all x of a neighborhood of \bar{x} , then \bar{x} is a strong local minimum of f .

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Actually, 1 is equivalent to $\nabla^2 f(\bar{x})$ being positive semidefinite and 2 – to f being positive definite.

Subdifferential characterization for a 2d order minorant

Worrying example: $f(x) = |x|^{\frac{3}{2}}$ – zero is a critical point and

$$\left| \frac{3}{2} \sqrt{|x|} \operatorname{sign} x + rx \right| \geq (\lambda + r)|x|$$

in a neighborhood of zero for any positive λ and r .

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Solution: Set for a closed $K \subset \mathbf{R}^n$

$$\Gamma_K(x) = \inf \{ \lambda \geq 0 : x \in \lambda K \}$$

(the a *gauge* or the *Minkowski function of K*) and

$$\Gamma_{\partial f}(x, y) = \Gamma_{\partial f(x)}(y).$$

Then *the condition $\Gamma_{\hat{\partial} f}(x, x - \bar{x}) \geq \alpha > 0$ is sufficient and, if f is semi-algebraic, the condition $\Gamma_{\partial f}(x, x - \bar{x}) \geq \alpha$ is necessary for f to have a quadratic minorant at \bar{x} .*

"First order" 2d order conditions

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Theorem 3. *Let f be a lsc function on \mathbf{R}^n finite at \bar{x} and $0 \in \partial f(\bar{x})$.*

1. If there are $\lambda > 0$ and $r > 0$ such that

$$d(0, \partial f(x) + r(x - \bar{x})) \geq (\lambda + r)\|x - \bar{x}\| \quad \& \quad \Gamma_{\hat{\partial}f}(x, x - \bar{x}) \geq \frac{1}{2r} > 0,$$

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1. If there are $\lambda > 0$ and $r > 0$ such that

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then \bar{x} is a strong local minimizer of f ;

2. If in addition f is semi-algebraic, then a necessary condition for f to have a local minimum at \bar{x} is the for every $r > 0$ the following properties hold:

- (a) the set-valued mapping $x \mapsto \partial f(x) + r(x - \bar{x})$ is strongly subregular at \bar{x} ;*
- (b) $\Gamma_{\partial f}(x, x - \bar{x}) \geq r^{-1}$ for all x of a neighborhood of \bar{x} .*

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THANK YOU

AND VERY BEST WISHES
TO TERRY