New Proofs of Maximal Monotonicity and Integrability of the Subdifferential of Convex Function

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Subdifferential: Maximal Monotonicity

Let X be a Banach space. For a proper, convex and lower semi-continuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ and $x \in \operatorname{dom} f$

$$\partial f(x) = \{p \in X^*;\ \langle p, y - x \rangle \le f(y) - f(x),\ \forall y \in X\}.$$

If $f(x) = \infty$ then $\partial f = \emptyset$.
By Definition

$$p \in \partial f(x),\, q \in \partial f(y) \Rightarrow \langle p - q, x - y \rangle\,.$$

That is, the multivalued map $x \rightrightarrows \partial f(x)$ from X to X^* is *monotone*. Such map is caled *maximal* monotone if it can not be properly extended while keeping monotonicity.
By Rockafellar Theorem ∂f is maximal monotone.

Subdifferential: Integrability

By Moreau-Rockafellar Theorem subdifferential operator is also *integrable* in the following sense:
If $\partial f \subset \partial g$ then $f = g + const$.
The proof of Moreau (which works only in reflexive Banach space) uses Lipschitz regularisations and transfer of the inclusion to the latter.
The one presented here shows the same ideas can work in general Banach space.

Tools: ε-Subdifferential

For $x \in \operatorname{dom} f$ and $\varepsilon > 0$:

$$\partial_\varepsilon f(x) = \{p \in X^*;\ \langle p, y - x\rangle \leq f(y) - f(x) + \varepsilon,\ \forall y \in X\}.$$

If $f(x) = \infty$ then $\partial_\varepsilon f = \emptyset$.

Theorem (Brøndsted-Rockafellar)

Let $f : X \to \mathbb{R} \cup \{+\infty\}$, be a proper, convex and lower semicontinuous function, let $\varepsilon > 0$ and $p \in \partial_\varepsilon f(x)$. Then there exists $q \in \partial f(z)$ such that

$$\|z - x\| \leq \sqrt{\varepsilon},\ \textit{and}\ \|q - p\| \leq \sqrt{\varepsilon}.$$

Partial case

Proposition

If f satisfies

$$\langle \partial f(x), x \rangle \geq 0, \quad \forall x \in X. \tag{1}$$

then $0 \in \partial f(0)$.

This partial case of Rockafellar's Theorem below implies the general case by an easy and well known reformulation [Phelps, Lecture Notes 1364, 1988 ed., p. 59].

Note that (1) can be viewed like minimality condition!

Theorem

Let X be a Banach space and let f be proper convex and lower semicontinuous. Then ∂f is maximal monotone mapping from X to X^*.

Let $(x_1, p) \in X \times X^*$ is monotone with respect to graph of f, that is,

$$\langle \partial f(x) - p, x - x_1 \rangle \geq 0, \quad \forall x \in X. \tag{2}$$

Consider

$$\bar{f}(x) := f(x + x_1) - p(x).$$

It is immediate to check that (2) implies (1) for $\bar{f}$. Therefore, by Proposition we get $0 \in \partial \bar{f}(0)$ which easily translates to $p \in \partial f(x_1)$. Therefore, ∂f could not be given proper and monotone extension.

f bounded below WLOG

Assume WLOG that $f(0) > 0$. By lower semicontinuity $f(\delta B_X) > 0$ for some $\delta > 0$. Since, by Hahn-Banach Theorem the sets $\mathrm{epi}\, f = \{(x,t) :\ t \geq f(x)\}$ and $\delta B_X \times \{0\}$ can be separated in $X \times \mathbb{R}$ by a bounded linear functional, there is $c > 0$ such that

$$f(x) > -c\|x\|, \quad \forall x \in X. \tag{3}$$

Consider for arbitrary $a \in (0, c)$

$$g_a(x) = \begin{cases} a\|x\|, & x \in B_X, \\ a + (c+1)\left(\|x\| - 1\right), & \|x\| > 1. \end{cases}$$

It is easy to check that g_a is Lipschitz and convex, and

$$\langle \partial g_a(x), x \rangle \geq a\|x\|, \quad \forall x \in X. \tag{4}$$

Consider

$$f_a := f + g_a.$$

By Sum Theorem

$$\langle \partial f_a(x), x\rangle \geq a\|x\|,\ \forall x \in X \Rightarrow \|\partial f_a(x)\| \geq a,\ \forall x \in X \setminus \{0\}. \quad (5)$$

From (3) it follows that f_a is bounded from below. Indeed, for $\|x\| > 1$ we have
$f_a(x) > -c\|x\| + a + (c+1)(\|x\| - 1) > \|x\| + a - c - 1.$
So, $\inf f_a \in \mathbb{R}$ and for each $\varepsilon > 0$

$$\varepsilon \mathrm{argmin}\, f_a := \{x :\ f_a(x) \leq \inf f_a + \varepsilon\}$$

is nonempty.

It is obvious by definition that

$$x \in \varepsilon \mathrm{argmin}\, f_a \iff 0 \in \partial_\varepsilon f_a(x).$$

By Brønstead-Rockafellar Theorem it follows that if $x \in \varepsilon \mathrm{argmin}\, f_a$ then there is $y \in x + \sqrt{\varepsilon} B_X$ such that $\partial f(y) \cap \sqrt{\varepsilon} B_{X^*} \neq \emptyset$.
Let $\{x_n\}$ be a minimisation sequence to f_a. There is another sequence $\{y_n\}$ with $p_n \in \partial f_a(y_n)$ such that

$$\lim_{n\to\infty} \|p_n\| = 0.$$

By (5) y_n's are eventually 0. By sraightforward limit in the definition of subdifferential, $0 \in \partial f_a(0)$.

Part 2: Integrability

For $n \in \mathbb{N}$ define Hausdorff regularization $\{f_n\}$ of f by

$$f_n(x) := \inf_{y \in X} \{f(y) + n\|x - y\|\}. \tag{6}$$

For sufficiently large n the functions f_n are finite valued.

$$M_\varepsilon^n(f; x) := \{y \in X : f(y) + n\|x - y\| \leq f_n(x) + \varepsilon\}.$$

Lemma

For any $\varepsilon \geq 0$, *and any* $y \in M^n_\varepsilon(f;x)$ *it holds that*

$$\partial f_n(x) \subset \partial_\varepsilon f(y) \cap \partial_\varepsilon n\| \cdot \|(x-y). \tag{7}$$

Let

$$\partial f(x) \subset \partial g(x), \quad \forall x \in X.$$

Define $\overline{f}(x) := f(x+\overline{x}) - \langle \overline{p}, x\rangle - f(\overline{x})$.
The function $\overline{f} : X \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lsc, $\overline{f}(0) = 0$ and $\partial\overline{f}(x) = \partial f(x+\overline{x}) - \overline{p}$. Hence, $0 \in \partial\overline{f}(0)$.
Similarly, $\overline{g}(x) := g(x+\overline{x}) - \langle \overline{p}, x\rangle - g(\overline{x})$.

Lemma

Let $\overline{f} : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lsc function such that $\overline{f}(0) = 0$ and $0 \in \partial \overline{f}(0)$. Let $s > 0$.
Then for $x \in B(0, s)$, $M^n_\varepsilon(\overline{f}, x) \subset B[0, 3s]$ for $n \geq 1/s$ and $\varepsilon \leq 1$.

Lemma

Let $\overline{f}, \overline{g} : X \to \mathbb{R} \cup \{+\infty\}$ be proper, convex and lsc functions such that $\overline{f}(0) = \overline{g}(0) = 0$, $0 \in \partial \overline{f}(0) \cap \partial \overline{g}(0)$, and $\partial \overline{f}(x) \subset \partial \overline{g}(x)$ for all $x \in X$. Let $s > 0$. Then for $n \geq 1/s$, $\partial \overline{f}_n(x) \subset \partial \overline{g}_n(x)$ for all $x \in B(0, s)$.

References

N. Zlateva, *Integrability through infimal regularization*, Comptes rendus des l'Academie bulgare des Sciences, 2015.