Positively α-far sets and existence results for generalized perturbed sweeping processes

A. Jourani and E. Vilches
Université de Bourgogne, Institut de Mathématiques de Bourgogne, UMR 5584 du CNRS, Dijon

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Motivations

Its importance comes from the study of problems in mechanics of elastoplastic materials, non-smooth mechanics, dynamics of systems with inelastic shocks, modeling and simulation of switched electrical circuits, crowd motion modeling, etc.

Position of the problem

$$\begin{cases} -\dot{x}(t) \in F(t, x(t)) + g(x(t))N\left(C(t), h(x(t))\right) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in h^{-1}\left(C(T_0)\right), \end{cases}$$

- X is a Hilbert space
- $N(C, x)$ is the Clarke's normal cone to C at $x \in C$
- $g\colon X \to \mathcal{L}(Y; X)$, $h\colon X \to Y$ are two mappings
- $F : [0, 1] \times X \rightrightarrows X$ is a **closed convex-valued** mapping
- $C : [T_0, T] \rightrightarrows X$ is a **closed-valued** mapping

Sweeping Process

$$\begin{cases} -\dot{x}(t) \in F(t, x(t)) + N\left(C(t), x(t)\right) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C(T_0) \end{cases}$$

Complementarity dynamical systems

A complementarity dynamical system (CDS) consists of ordinary differential equations coupled to complementarity conditions which can be specified by functions $F\colon [T_0, T] \times X \to Y$, $g\colon X \to \mathcal{L}(Y; X)$ and $H\colon X \to Y$. The defining equations for the CDS corresponding to F, g and H are

$$\begin{aligned} \dot{x}(t) &= F(t, x(t)) + g(x(t))u(t), \\ y(t) &= H(t, x(t), u(t)), \\ K \ni & y(t) \perp u(t) \in K^*, \end{aligned} \tag{1}$$

where $K \subset Y$ is a closed convex cone and $K^* = \{d \in Y \colon \langle v, d \rangle \geq 0 \, \forall v \in K\}$ denotes the dual cone of K.

Basic assumptions

(H_F):

- For each $x \in X$, $F_i(\cdot, x)$ is measurable.
- For a.e. $t \in [T_0, T]$, $F_i(t, \cdot)$ is upper semicontinuous from X into X_w.
- There exist $\alpha, \beta \in L^1(T_0, T)$ such that

$$\sup\{\|w\| \colon w \in F(t,x)\} \leq \alpha(t)\|x\| + \beta(t) \, \forall x \in X, \text{ a.e. } t \in [T_0, T].$$

(H_C): The set $C(t)$ varies in an absolutely continuous way, that is, there exists an absolutely continuous non negative function $\zeta : [0, T] \to \mathbb{R}_+$ such that

$$|d(x, C(t)) - d(y, C(s))| \leq \|x - y\| + |\zeta(t) - \zeta(s)|$$

for all $x, y \in X$ and all $s, t \in [0, T]$

Basic assumptions (continued)

$(H_{(g,h)})$:

- The function g is continuous and there exist $M_1 \geq 0$ and $M_2 > 0$;

$$\|g(x)\| \leq M_1\|x\| + M_2 \quad \text{for all } x \in X,$$

- $h\colon X \to Y$ is a differentiable function with $\sup\limits_{x\in X} \|Dh(x)\| \leq M$, with $M > 0$.
- There exists $\lambda > 0$ such that for all $x \in X$

$$\langle Dh(x)g(x)y, y\rangle \geq \lambda\|y\|^2 \, \forall y \in Y.$$

Sweeping process : Existence

- Convex sets in Hilbert spaces : Moreau, Henry,...

$$\dot{u}(t) \in -N(C(t), u(t)) + F(u(t)) \text{ a.e.t}$$

- Regular sets in finite dimension : Cornet...

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- Closed sets in finite dimension : Benabdellah, Colombo-Goncharov, Thibault, Valadier,...
- Uniformly prox-regular sets in Hilbert spaces : Edmond-Thibault, Faik-Syam, Colombo-Goncharov, ...
- Uniformly subsmooth sets in Hilbert spaces: Noel-Thibault

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Theorem

Let $A\colon X \to X$ be a single valued maximal monotone operator such that $-A$ generates a compact semigroup on $\overline{D(A)} := X$ and let $F\colon [T_0, T] \times \overline{D(A)} \rightrightarrows X$ be a set-valued map with closed and convex values satisfying

- *For each $x \in X$, $F(\cdot, x)$ is measurable.*
- *For a.e. $t \in [T_0, T]$, $F(t, \cdot)$ is upper semicontinuous from X into X_w.*
- *There exist $\gamma, \delta \in L^1(T_0, T)$ such that*

$$\inf\{\|w\|\colon w \in F(t,x)\} \leq \gamma(t)\|x\| + \delta(t) \quad a.e. \ t \in [T_0, T], x \in X.$$

Then there exists at least one solution of

$$\begin{cases} \dfrac{dx}{dt}(t) \in -Ax(t) + F(t, x(t)) & a.e. \ t \in [T_0, T], \\ x(T_0) = x_0, \end{cases}$$

Positively α-far sets

Definition

Let $\alpha \in (0,1]$ and $\rho \in (0,+\infty]$. Let S be a nonempty closed subset of X with $S \neq X$. We say that the Clarke subdifferential of the distance function $d(\cdot, S)$ keeps the origin α-far-off on the open ρ-tube around S provided

$$\alpha \leq \inf_{x \in U_\rho(S)} d(0, \partial d(x,S)). \tag{2}$$

Also, if E is a given nonempty set, we say that the family $(S(t))_{t\in E}$ is positively α-far if every set $S(t)$ satisfies (2) with the same $\alpha > 0$ and $\rho > 0$.

Recall that the Clarke subdifferential of the distance can be characterized via the formula:

$$\partial d(x,S) = \bigcap_{\gamma>0} \overline{\mathrm{co}}\left(\frac{x - P_S^\gamma(x)}{d_S(x)}\right), \tag{3}$$

where $P_S^\gamma(x) = \{z \in S \colon \|x - z\| \leq d(x,S) + \gamma\}$.

Why are they needed?

$$-\dot{u}(t) \in \frac{1}{\alpha^2}|\dot{\zeta}(t)|\,\partial d(u(t), C(t)) \text{ a.e. } t \tag{4}$$

$$\begin{cases} -\dot{u}(t) \in \frac{1}{\alpha^2}|\dot{\zeta}(t)|\,\partial d(u(t), C(t)) \text{ a.e. } t \\ u(0) \in C(0). \end{cases} \tag{5}$$

$$\begin{cases} -\dot{x}(t) \in N(C(t), x(t)) \quad \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C(T_0) \end{cases} \tag{6}$$

A simple example

$C(t) = \{x \in \mathbb{R}^2 : \|x\|_2 = t\}$. The arc $u(t) = 0$ for all t, is a solution of the differential inclusion (5), while $0 \notin C(t)$, for all $t > 0$. Hence u is not a solution of (6).

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Feasibility in finite time and reduction theorem

Theorem (Flåm, Hiriart-Urruty, J.)

Suppose that x is a solution of (4) and the family $(C(t))_t$ is positively $\alpha-$far on $U_\rho(S)$. Then there exists $\bar{t} \in [0, T]$ such that for all $t \geq \bar{t}$, $x(t) \in C(t)$.

Theorem (Haddad, Thibault, J.)

Assume that the family $(C(t))_t$ is positively $\alpha-$far on $U_\rho(S)$. Then any absolutely continuous solution $u : [0, T] \to H$ of (5) is a solution of (6).

- S.D. Flåm, J. B. Hiriart-Urruty, A. Jourani, *Feasibility in finite time*, JDCS (2009), 537-555.
- T. Haddad, A. Jourani, L. Thibault, *Reduction of sweeping process to unconstrained differential inclusion*, PJO (2008), 493-512.

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Proposition

Let S be a closed subset of X and $\rho > 0$.

1. *Assume that the following property holds: For every $x \in U_\rho(S)$ there exists $\gamma(x)$ such that for all $\gamma \in]0, \gamma(x)[$*

$$u_1^*, u_2^* \in \frac{x - P_S^\gamma(x)}{d(x,S)} \quad \Rightarrow \quad \langle u_1^*, u_2^* \rangle \geq \alpha^2 + \theta(\gamma, x), \tag{7}$$

   where $\lim_{\gamma \downarrow 0} \theta(\gamma, x) = 0$ for all $x \in U_\rho(S)$. Then the origin is kept positively α-far from the Clarke subdifferential of the distance function $d(\cdot, S)$ on $U_\rho(S)$.

2. *Assume that the origin is kept positively α-far from the Clarke subdifferential of the distance function $d(\cdot, S)$ on $U_\rho(S)$. Then the following property holds:*

$$\forall x \in U_\rho(S) \quad u_1^*, u_2^* \in \partial d(x,S) \quad \Rightarrow \quad \langle u_1^*, u_2^* \rangle \geq 2\alpha^2 - 1. \tag{8}$$

Proposition

Let $S \subset X$ *be a closed set,* $\alpha \in]0,1[$ *and* $\rho > 0$.

1. *If the origin is kept positively* α*-far from the Clarke subdifferential of the distance function* $d(\cdot, S)$ *on* $U_\rho(S)$*, then*

$$\forall x \in U_\rho(S), \forall x^* \in \partial d(x,S); \quad d(x - x^* d(x,S), S) \leq d(x,S)\sqrt{1-\alpha^2}. \tag{9}$$

2. *Conversely, if (9) is satisfied then the origin is kept positively* $1-\sqrt{1-\alpha^2}$*-far from the Clarke subdifferential of the distance function* $d(\cdot, S)$ *on* $U_\rho(S)$.

Proposition

Let S be a closed subset of X and let $\rho > 0$ and $\alpha > 0$. Then the following assertions are equivalent:

1. *The origin is kept positively α-far from the Clarke subdifferential of the distance function $d(\cdot, S)$ on $U_\rho(S)$.*
2. *For all $\varepsilon \in]0, \alpha[$, there exists a locally Lipschitz function $V : X \setminus S \mapsto X$ such that: $\forall x \in U_\rho(S)$, $\|V(x)\| \leq 1 + \varepsilon$,*

$$\forall \gamma > 0, \quad \inf_{u \in P_S^\gamma(x)} \langle x - u, V(x) \rangle \geq (\alpha - \varepsilon) d(x, S).$$

Paraconvex sets

Following Michael, a set S is α-paraconvex for some $\alpha \in [0,1[$ if, whenever $x \in X$ and $r > 0$ are such that $d(x,S) < r$, then

$$d(u,S) \leq \alpha r \quad \forall\, u \in \mathrm{co}[B(x,r) \cap S].$$

This implies immediately that

$$\forall \rho > 0,\ \forall x \in U_\rho(S),\ \forall \gamma > 0, \quad \alpha(d(x,S)+\gamma) \geq d(u,S)\, \forall u \in \mathrm{co} P_S^{\gamma}(x). (10)$$

Proposition

Let $S \subset X$ be a closed set which is α-paraconvex for some $\alpha \in [0,1[$. Then for each $\rho > 0$, the Clarke subdifferential of the distance function $d(\cdot,S)$ keeps the origin $(1-\alpha)$-far-off on the open ρ-tube around S.

Prox-regular sets

Let S be a closed set and $\rho > 0$. We recall that S is ρ-uniformly prox-regular if for all $x \in S$ and all $v \in N(S, x)$ with $\|v\| < 1$, x is the unique nearest point of S to $x + \rho^{-1}v$, i.e.,

$$x = P_S(x + \rho^{-1}v).$$

Here $P_S(u)$ denotes the unique nearest point of S to u.

Proposition

Let S be a closed subset of X with $S \neq X$ and $\rho > 0$. Then S is ρ-uniformly prox-regular if and only if the origin is kept positively 1-far from the Clarke subdifferential of the distance function $d(\cdot, S)$ on $U_\rho(S)$.

Uniformly subsmooth sets

C is *uniformly subsmooth*, if for every $\varepsilon > 0$ there exists $\delta > 0$, such that

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\varepsilon \|x_1 - x_2\|, \tag{11}$$

holds for all $x_1, x_2 \in C$ satisfying $\|x_1 - x_2\| < \delta$ and all $x_i^* \in N(C, x_i) \cap \mathbb{B}_X$ for $i = 1, 2$.

Proposition

Assume that S is uniformly subsmooth. Then, for all $\varepsilon \in]0,1[$ there exists $\rho \in]0, +\infty[$ such that the origin is kept positively $\sqrt{1-\varepsilon}$-far from the Clarke subdifferential of the distance function $d(\cdot, S)$ on the open ρ-tube $U_\rho(S) = \{y \in H \colon 0 < d(y,S) < \rho\}$, i.e.,

$$\sqrt{1-\varepsilon} \leq \inf_{y \in U_\rho(S)} d\left(0, \partial d(y,S)\right).$$

Generalized sweeping process

$$\begin{cases} -\dot{x}(t) \in F(t, x(t)) + g(x(t)) N\left(C(t), h(x(t))\right) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in h^{-1}\left(C(T_0)\right), \end{cases} \tag{12}$$

Theorem

Suppose that the following assumption holds true:

$(\mathcal{H}_{(h^{-1}(C))})$:
For all $r > 0$ and $t \in [T_0, T]$ the set $h^{-1}\left(C(t)\right) \cap r\mathbb{B}$ is compact in X.

Then there exists at least one solution of (12) satisfying

$$\|\dot{x}(t)\| \leq \dot{\sigma}(t)\left(\|x_0\| + \int_{T_0}^{t} \beta(s)ds\right) + M_2\dot{\sigma}(t)\int_{T_0}^{t} \mu(s)ds + \beta(t) + M_2\mu(t)$$

where η, σ and μ are functions from $[T_0, T]$ to $\mathbb{R}_+$ defined by the following formulas:

$$\eta(t) = M\dot{\sigma}(t)\left(\|x_0\| + \int_{T_0}^{t} \beta(s)ds\right) + M\beta(t) + |\dot{\zeta}(t)|,$$

$$\sigma(t) = \exp\left(\int_{T_0}^{t} (\alpha(s) + M_1)\, ds\right),$$

$$\mu(t) = MM_2\frac{\eta(t)}{\alpha_0^2\lambda} + MM_2\frac{\dot{\sigma}(t)}{\alpha_0^2\lambda}\int_{T_0}^{t} \frac{MM_2\eta(s)}{\alpha_0^2\lambda}\exp\left(MM_2\int_{s}^{t}\frac{\dot{\sigma}(\tau)}{\alpha_0^2\lambda}d\tau\right) ds.$$

Sweeping Process

Corollary

Suppose that the following assumption holds true:

- *the family* $(C(t))_{t\in[T_0,T]}$ *is positively* α*-far*
- *For all* $r > 0$ *and* $t \in [T_0, T]$ *the set* $C(t) \cap r\mathbb{B}$ *is compact in* X.

Then there exists at least one solution of the following sweeping process:

$$\begin{cases} -\dot{x}(t) \in F(t, x(t)) + N(C(t), x(t)) & a.e.\ t \in [T_0, T], \\ x(T_0) = x_0 \in C(T_0) \end{cases}$$

Complementarity dynamical systems

A complementarity dynamical system (CDS) consists of ordinary differential equations coupled to complementarity conditions which can be specified by functions $F\colon [T_0, T] \times X \to Y$, $g\colon X \to \mathcal{L}(Y; X)$ and $H\colon X \to Y$. The defining equations for the CDS corresponding to F, g and H are

$$\begin{aligned} \dot{x}(t) &= F(t, x(t)) + g(x(t))u(t), \\ y(t) &= H(t, x(t), u(t)), \\ K \ni y(t) &\perp u(t) \in K^*, \end{aligned} \tag{13}$$

where $K \subset Y$ is a closed convex cone and $K^* = \{d \in Y \colon \langle v, d\rangle \geq 0 \, \forall v \in K\}$ denotes the dual cone of K and $H(t, x(t), u(t)) = h(x(t)) + d(t)$ where $h\colon X \to Y$ is a differentiable function and $d\colon [T_0, T] \to Y$ is an absolutely continuous function.

Theorem

Assume that F satisfies (H_F) and K is ball-compact. Then for every x_0 with $h(x_0)+d(T_0)\in K$ there exists at least one absolutely continuous solution x of (13) satisfying $x(T_0)=x_0$ and

$$\|\dot{x}(t)\| \leq \dot{\sigma}(t)\left(\|x_0\| + \int_{T_0}^{t}\beta(s)ds\right) + M_2\dot{\sigma}(t)\int_{T_0}^{t}\mu(s)ds + \beta(t) + M_2\mu(t)$$

where η, σ and μ are functions from $[T_0,T]$ to $\mathbb{R}_+$ defined by the following formulas:

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$$\sigma(t) = \exp\left(\int_{T_0}^{t}(\alpha(s)+M_1)\,ds\right),$$

$$\mu(t) = MM_2\frac{\eta(t)}{\lambda} + MM_2\frac{\dot{\sigma}(t)}{\lambda}\int_{T_0}^{t}\frac{MM_2\eta(s)}{\lambda}\exp\left(MM_2\int_{s}^{t}\frac{\dot{\sigma}(\tau)}{\lambda}d\tau\right)ds.$$

Gradient complementarity dynamical systems (GCDS)

The gradient complementarity dynamical systems (GCDS) corresponds to the particular case where

$$g \equiv [Dh]^* .$$

$(H_(h, K))$:
The mapping h is differentiable with uniformly continuous derivative Dh, $g \equiv [Dh]^*$ and there exists $k > 0$ such that

$$\mathbb{B}_Y \subset Dh(x)k\mathbb{B}_X - K \quad \text{for all } x \in h^{-1}(K).$$

Theorem

Assume that F satisfies (H_F), K is bal-compact and $(H_(h, K))$ holds. Then, for every x_0 with $h(x_0) + d(T_0) \in K$, there exists at least one absolutely continuous solution x of (13) satisfying $x(T_0) = x_0$ and

$$\|\dot{x}(t)\| \leq \frac{1+\alpha_0^2}{\alpha_0^2}\left(\eta(t) + \dot{\sigma}(t)\int_{T_0}^{t}\frac{\eta(s)}{\alpha_0^2}\exp\left(\frac{\sigma(t)}{\alpha_0^2} - \frac{\sigma(s)}{\alpha_0^2}\right)ds\right) - k'|\dot{d}(t)|$$

where η and σ are functions from $[T_0, T]$ to $\mathbb{R}_+$ defined by the following formulas:

$$\eta(t) = \dot{\sigma}(t)\left(\|x_0\| + \int_{T_0}^{t}\beta(s)ds\right) + \beta(t) + k'|\dot{d}(t)|,$$

$$\sigma(t) = \exp\left(\int_{T_0}^{t}\alpha(s)ds\right).$$

- H. Benabdellah, Existence of solutions to nonconvex sweeping process, JDE (2000), 286-295.
- J. M. Borwein and H. M. Strojwas, Proximal analysis and boundaries of closed sets in Banach space, Can. J. Math. (1987), 428-472.
- M. Bounkhel and L. Thibault, Nonconvex sweeping process and prox-regularity in Hilbert space, JNCA, 359-374.
- C. Castaing, T.X. Duc Ha and M. Valadier, Evolution equations governed by sweeping process, Set-Valued Anal. (1993), 109-139.
- B. Cornet, Existence of slow solutions for a class of differential inclusions, JMAA, (1973), 179-186.
- G. Colombo and V.V. Goncharov, The sweeping process without convexity, Set-Valued Anal. (1999), 357-374.
- J.F. Edmond, L. Thibault, BV solution of nonconvex sweeping process with perturbation, JDE (2006), 135-179.

- L.A. Faik, A. Syam, Differential inclusions governed by a nonconvex sweeping process, JNCA, 2 (2001), 381-392.
- S.D. Flåm, J. B. Hiriart-Urruty, A. Jourani, *Feasibility in finite time*, JDCS (2009), 537-555.
- T. Haddad, A. Jourani, L. Thibault, *Reduction of sweeping process to unconstrained differential inclusion*, PJO (2008), 493-512.
- J. B. Hiriart-Urruty, Tangent Cones, Generalized Gradients and Mathematical Programming in Banach Spaces, MOR (1979), pp. 79-97
- M.D.P.Monteiro Marques, Differential inclusions in Nonsmooth Mechanical Problems, Shocks and Dry Friction. Birkhäuser, Basel, (1993).
- J.J. Moreau, Evolution problem associated with a moving convex set in a Hilbert space, JDE (1977), 347-374.