

Positively α -far sets and existence results for generalized perturbed sweeping processes

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Motivations

Its importance comes from the study of problems in mechanics of elastoplastic materials, non-smooth mechanics, dynamics of systems with inelastic shocks, modeling and simulation of switched electrical circuits, crowd motion modeling, etc.

Position of the problem

$$\begin{cases} -\dot{x}(t) \in F(t, x(t)) + g(x(t))N(C(t), h(x(t))) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in h^{-1}(C(T_0)), \end{cases}$$

- X is a Hilbert space
- $N(C, x)$ is the Clarke's normal cone to C at $x \in C$
- $g: X \rightarrow \mathcal{L}(Y; X)$, $h: X \rightarrow Y$ are two mappings
- $F: [0, 1] \times X \rightrightarrows X$ is a **closed convex-valued** mapping
- $C: [T_0, T] \rightrightarrows X$ is a **closed-valued** mapping

Sweeping Process

$$\begin{cases} -\dot{x}(t) \in F(t, x(t)) + N(C(t), x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C(T_0) \end{cases}$$

Complementarity dynamical systems

A complementarity dynamical system (CDS) consists of ordinary differential equations coupled to complementarity conditions which can be specified by functions $F: [T_0, T] \times X \rightarrow Y$, $g: X \rightarrow \mathcal{L}(Y; X)$ and $H: X \rightarrow Y$. The defining equations for the CDS corresponding to F , g and H are

$$\begin{aligned}\dot{x}(t) &= F(t, x(t)) + g(x(t))u(t), \\ y(t) &= H(t, x(t), u(t)), \\ K \ni y(t) &\perp u(t) \in K^*,\end{aligned}\tag{1}$$

where $K \subset Y$ is a closed convex cone and $K^* = \{d \in Y: \langle v, d \rangle \geq 0 \forall v \in K\}$ denotes the dual cone of K .

Basic assumptions

(H_F) :

- For each $x \in X$, $F_i(\cdot, x)$ is measurable.
- For a.e. $t \in [T_0, T]$, $F_i(t, \cdot)$ is upper semicontinuous from X into X_w .
- There exist $\alpha, \beta \in L^1(T_0, T)$ such that

$$\sup\{\|w\| : w \in F(t, x)\} \leq \alpha(t)\|x\| + \beta(t) \forall x \in X, \text{ a.e. } t \in [T_0, T].$$

(H_C) : The set $C(t)$ varies in an absolutely continuous way, that is, there exists an absolutely continuous non negative function $\zeta : [0, T] \rightarrow \mathbb{R}_+$ such that

$$|d(x, C(t)) - d(y, C(s))| \leq \|x - y\| + |\zeta(t) - \zeta(s)|$$

for all $x, y \in X$ and all $s, t \in [0, T]$

Basic assumptions (continued)

$(H_{(g,h)})$:

- The function g is continuous and there exist $M_1 \geq 0$ and $M_2 > 0$;

$$\|g(x)\| \leq M_1\|x\| + M_2 \quad \text{for all } x \in X,$$

- $h: X \rightarrow Y$ is a differentiable function with $\sup_{x \in X} \|Dh(x)\| \leq M$, with $M > 0$.
- There exists $\lambda > 0$ such that for all $x \in X$

$$\langle Dh(x)g(x)y, y \rangle \geq \lambda\|y\|^2 \quad \forall y \in Y.$$

Sweeping process : Existence

- Convex sets in Hilbert spaces : Moreau, Henry,...

$$\dot{u}(t) \in -N(C(t), u(t)) + F(u(t)) \text{ a.e. } t$$

- Regular sets in finite dimension : Cornet...

$$\dot{u}(t) \in -N(C, u(t)) + F(u(t)) \text{ a.e. } t$$

- Convex sets or complement of open convex sets :
Castaing-Ha-Valadier,...

$$\dot{u}(t) \in -N(C(t), u(t)) + F(t, u(t)) \text{ a.e. } t$$

- Closed sets in finite dimension : Benabdellah, Colombo-Goncharov, Thibault, Valadier,...
- Uniformly prox-regular sets in Hilbert spaces : Edmond-Thibault, Faik-Syam, Colombo-Goncharov, ...
- Uniformly subsmooth sets in Hilbert spaces: Noel-Thibault

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Theorem

Let $A: X \rightarrow X$ be a single valued maximal monotone operator such that $-A$ generates a compact semigroup on $\overline{D(A)} := X$ and let $F: [T_0, T] \times \overline{D(A)} \rightrightarrows X$ be a set-valued map with closed and convex values satisfying

- For each $x \in X$, $F(\cdot, x)$ is measurable.
- For a.e. $t \in [T_0, T]$, $F(t, \cdot)$ is upper semicontinuous from X into X_w .
- There exist $\gamma, \delta \in L^1(T_0, T)$ such that

$$\inf\{\|w\|: w \in F(t, x)\} \leq \gamma(t)\|x\| + \delta(t) \quad \text{a.e. } t \in [T_0, T], x \in X.$$

Then there exists at least one solution of

$$\begin{cases} \frac{dx}{dt}(t) \in -Ax(t) + F(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0, \end{cases}$$

Positively α -far sets

Definition

Let $\alpha \in (0, 1]$ and $\rho \in (0, +\infty]$. Let S be a nonempty closed subset of X with $S \neq X$. We say that the Clarke subdifferential of the distance function $d(\cdot, S)$ keeps the origin α -far-off on the open ρ -tube around S provided

$$\alpha \leq \inf_{x \in U_\rho(S)} d(0, \partial d(x, S)). \quad (2)$$

Also, if E is a given nonempty set, we say that the family $(S(t))_{t \in E}$ is positively α -far if every set $S(t)$ satisfies (2) with the same $\alpha > 0$ and $\rho > 0$.

Recall that the Clarke subdifferential of the distance can be characterized via the formula:

$$\partial d(x, S) = \bigcap_{\gamma > 0} \overline{\text{co}} \left(\frac{x - P_S^\gamma(x)}{d_S(x)} \right), \quad (3)$$

where $P_S^\gamma(x) = \{z \in S : \|x - z\| \leq d(x, S) + \gamma\}$.

Why are they needed?

$$- \dot{u}(t) \in \frac{1}{\alpha^2} |\dot{C}(t)| \partial d(u(t), C(t)) \quad \text{a.e. } t \quad (4)$$

$$\begin{cases} -\dot{u}(t) \in \frac{1}{\alpha^2} |\dot{C}(t)| \partial d(u(t), C(t)) \quad \text{a.e. } t \\ u(0) \in C(0). \end{cases} \quad (5)$$

$$\begin{cases} -\dot{x}(t) \in N(C(t), x(t)) \quad \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C(T_0) \end{cases} \quad (6)$$

A simple example

$C(t) = \{x \in \mathbb{R}^2 : \|x\|_2 = t\}$. The arc $u(t) = 0$ for all t , is a solution of the differential inclusion (5), while $0 \notin C(t)$, for all $t > 0$. Hence u is not a solution of (6).

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Feasibility in finite time and reduction theorem

Theorem (Flåm, Hiriart-Urruty, J.)

Suppose that x is a solution of (4) and the family $(C(t))_t$ is positively α -far on $U_\rho(S)$. Then there exists $\bar{t} \in [0, T]$ such that for all $t \geq \bar{t}$, $x(t) \in C(t)$.

Theorem (Haddad, Thibault, J.)

Assume that the family $(C(t))_t$ is positively α -far on $U_\rho(S)$. Then any absolutely continuous solution $u : [0, T] \rightarrow H$ of (5) is a solution of (6).

- S.D. Flåm, J. B. Hiriart-Urruty, A. Jourani, *Feasibility in finite time*, JDCS (2009), 537-555.
- T. Haddad, A. Jourani, L. Thibault, *Reduction of sweeping process to unconstrained differential inclusion*, PJO (2008), 493-512.

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Proposition

Let S be a closed subset of X and $\rho > 0$.

- 1 Assume that the following property holds: For every $x \in U_\rho(S)$ there exists $\gamma(x)$ such that for all $\gamma \in]0, \gamma(x)[$

$$u_1^*, u_2^* \in \frac{x - P_S^\gamma(x)}{d(x, S)} \Rightarrow \langle u_1^*, u_2^* \rangle \geq \alpha^2 + \theta(\gamma, x), \quad (7)$$

where $\lim_{\gamma \downarrow 0} \theta(\gamma, x) = 0$ for all $x \in U_\rho(S)$. Then the origin is kept positively α -far from the Clarke subdifferential of the distance function $d(\cdot, S)$ on $U_\rho(S)$.

- 2 Assume that the origin is kept positively α -far from the Clarke subdifferential of the distance function $d(\cdot, S)$ on $U_\rho(S)$. Then the following property holds:

$$\forall x \in U_\rho(S) \quad u_1^*, u_2^* \in \partial d(x, S) \Rightarrow \langle u_1^*, u_2^* \rangle \geq 2\alpha^2 - 1. \quad (8)$$

Proposition

Let $S \subset X$ be a closed set, $\alpha \in]0, 1[$ and $\rho > 0$.

- 1 If the origin is kept positively α -far from the Clarke subdifferential of the distance function $d(\cdot, S)$ on $U_\rho(S)$, then

$$\forall x \in U_\rho(S), \forall x^* \in \partial d(x, S); \quad d(x - x^* d(x, S), S) \leq d(x, S) \sqrt{1 - \alpha^2}. \quad (9)$$

- 2 Conversely, if (9) is satisfied then the origin is kept positively $1 - \sqrt{1 - \alpha^2}$ -far from the Clarke subdifferential of the distance function $d(\cdot, S)$ on $U_\rho(S)$.

Proposition

Let S be a closed subset of X and let $\rho > 0$ and $\alpha > 0$. Then the following assertions are equivalent:

- 1 The origin is kept positively α -far from the Clarke subdifferential of the distance function $d(\cdot, S)$ on $U_\rho(S)$.
- 2 For all $\varepsilon \in]0, \alpha[$, there exists a locally Lipschitz function $V : X \setminus S \mapsto X$ such that: $\forall x \in U_\rho(S), \|V(x)\| \leq 1 + \varepsilon$,

$$\forall \gamma > 0, \inf_{u \in P_S^\gamma(x)} \langle x - u, V(x) \rangle \geq (\alpha - \varepsilon)d(x, S).$$

Paraconvex sets

Following Michael, a set S is α -paraconvex for some $\alpha \in [0, 1[$ if, whenever $x \in X$ and $r > 0$ are such that $d(x, S) < r$, then

$$d(u, S) \leq \alpha r \quad \forall u \in \text{co}[B(x, r) \cap S].$$

This implies immediately that

$$\forall \rho > 0, \forall x \in U_\rho(S), \forall \gamma > 0, \quad \alpha(d(x, S) + \gamma) \geq d(u, S) \quad \forall u \in \text{co}P_S^\gamma(x). \quad (10)$$

Proposition

Let $S \subset X$ be a closed set which is α -paraconvex for some $\alpha \in [0, 1[$. Then for each $\rho > 0$, the Clarke subdifferential of the distance function $d(\cdot, S)$ keeps the origin $(1 - \alpha)$ -far-off on the open ρ -tube around S .

Prox-regular sets

Let S be a closed set and $\rho > 0$. We recall that S is ρ -uniformly prox-regular if for all $x \in S$ and all $v \in N(S, x)$ with $\|v\| < 1$, x is the unique nearest point of S to $x + \rho^{-1}v$, i.e.,

$$x = P_S(x + \rho^{-1}v).$$

Here $P_S(u)$ denotes the unique nearest point of S to u .

Proposition

Let S be a closed subset of X with $S \neq X$ and $\rho > 0$. Then S is ρ -uniformly prox-regular if and only if the origin is kept positively 1-far from the Clarke subdifferential of the distance function $d(\cdot, S)$ on $U_\rho(S)$.

Uniformly subsmooth sets

C is *uniformly subsmooth*, if for every $\varepsilon > 0$ there exists $\delta > 0$, such that

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\varepsilon \|x_1 - x_2\|, \quad (11)$$

holds for all $x_1, x_2 \in C$ satisfying $\|x_1 - x_2\| < \delta$ and all $x_i^* \in N(C, x_i) \cap \mathbb{B}_X$ for $i = 1, 2$.

Proposition

Assume that S is uniformly subsmooth. Then, for all $\varepsilon \in]0, 1[$ there exists $\rho \in]0, +\infty[$ such that the origin is kept positively $\sqrt{1 - \varepsilon}$ -far from the Clarke subdifferential of the distance function $d(\cdot, S)$ on the open ρ -tube $U_\rho(S) = \{y \in H : 0 < d(y, S) < \rho\}$, i.e.,

$$\sqrt{1 - \varepsilon} \leq \inf_{y \in U_\rho(S)} d(0, \partial d(y, S)).$$

Generalized sweeping process

$$\begin{cases} -\dot{x}(t) \in F(t, x(t)) + g(x(t))N(C(t), h(x(t))) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in h^{-1}(C(T_0)), \end{cases} \quad (12)$$

Theorem

Suppose that the following assumption holds true:

$(H_{(h^{-1}(C))})$:

For all $r > 0$ and $t \in [T_0, T]$ the set $h^{-1}(C(t)) \cap r\mathbb{B}$ is compact in X .

Then there exists at least one solution of (12) satisfying

$$\|\dot{x}(t)\| \leq \dot{\sigma}(t) \left(\|x_0\| + \int_{T_0}^t \beta(s) ds \right) + M_2 \dot{\sigma}(t) \int_{T_0}^t \mu(s) ds + \beta(t) + M_2 \mu(t)$$

where η , σ and μ are functions from $[T_0, T]$ to \mathbb{R}_+ defined by the following formulas:

$$\eta(t) = M\dot{\sigma}(t) \left(\|x_0\| + \int_{T_0}^t \beta(s) ds \right) + M\beta(t) + |\dot{\zeta}(t)|,$$

$$\sigma(t) = \exp \left(\int_{T_0}^t (\alpha(s) + M_1) ds \right),$$

$$\mu(t) = MM_2 \frac{\eta(t)}{\alpha_0^2 \lambda} + MM_2 \frac{\dot{\sigma}(t)}{\alpha_0^2 \lambda} \int_{T_0}^t \frac{MM_2 \eta(s)}{\alpha_0^2 \lambda} \exp \left(MM_2 \int_s^t \frac{\dot{\sigma}(\tau)}{\alpha_0^2 \lambda} d\tau \right) ds.$$

Sweeping Process

Corollary

Suppose that the following assumption holds true:

- the family $(C(t))_{t \in [T_0, T]}$ is positively α -far
- For all $r > 0$ and $t \in [T_0, T]$ the set $C(t) \cap r\mathbb{B}$ is compact in X .

Then there exists at least one solution of the following sweeping process:

$$\begin{cases} -\dot{x}(t) \in F(t, x(t)) + N(C(t), x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C(T_0) \end{cases}$$

Complementarity dynamical systems

A complementarity dynamical system (CDS) consists of ordinary differential equations coupled to complementarity conditions which can be specified by functions $F: [T_0, T] \times X \rightarrow Y$, $g: X \rightarrow \mathcal{L}(Y; X)$ and $H: X \rightarrow Y$. The defining equations for the CDS corresponding to F , g and H are

$$\begin{aligned}\dot{x}(t) &= F(t, x(t)) + g(x(t))u(t), \\ y(t) &= H(t, x(t), u(t)), \\ K \ni y(t) \perp u(t) &\in K^*,\end{aligned}\tag{13}$$

where $K \subset Y$ is a closed convex cone and

$K^* = \{d \in Y: \langle v, d \rangle \geq 0 \forall v \in K\}$ denotes the dual cone of K and $H(t, x(t), u(t)) = h(x(t)) + d(t)$ where $h: X \rightarrow Y$ is a differentiable function and $d: [T_0, T] \rightarrow Y$ is an absolutely continuous function.

Theorem

Assume that F satisfies (H_F) and K is ball-compact. Then for every x_0 with $h(x_0) + d(T_0) \in K$ there exists at least one absolutely continuous solution x of (13) satisfying $x(T_0) = x_0$ and

$$\|\dot{x}(t)\| \leq \dot{\sigma}(t) \left(\|x_0\| + \int_{T_0}^t \beta(s) ds \right) + M_2 \dot{\sigma}(t) \int_{T_0}^t \mu(s) ds + \beta(t) + M_2 \mu(t)$$

where η , σ and μ are functions from $[T_0, T]$ to \mathbb{R}_+ defined by the following formulas:

$$\eta(t) = M \dot{\sigma}(t) \left(\|x_0\| + \int_{T_0}^t \beta(s) ds \right) + M \beta(t) + |\dot{d}(t)|,$$

$$\sigma(t) = \exp \left(\int_{T_0}^t (\alpha(s) + M_1) ds \right),$$

$$\mu(t) = MM_2 \frac{\eta(t)}{\lambda} + MM_2 \frac{\dot{\sigma}(t)}{\lambda} \int_{T_0}^t \frac{MM_2 \eta(s)}{\lambda} \exp \left(MM_2 \int_s^t \frac{\dot{\sigma}(\tau)}{\lambda} d\tau \right) ds.$$

Gradient complementarity dynamical systems (GCDS)

The gradient complementarity dynamical systems (GCDS) corresponds to the particular case where

$$g \equiv [Dh]^* .$$

$(H(h, K))$:

The mapping h is differentiable with uniformly continuous derivative Dh , $g \equiv [Dh]^*$ and there exists $k > 0$ such that

$$\mathbb{B}_Y \subset Dh(x)k\mathbb{B}_X - K \quad \text{for all } x \in h^{-1}(K) .$$

Theorem








Assume that F satisfies (H_F) , K is bal-compact and $(H(h, K))$ holds. Then, for every x_0 with $h(x_0) + d(T_0) \in K$, there exists at least one absolutely continuous solution x of (13) satisfying $x(T_0) = x_0$ and







$$\|\dot{x}(t)\| \leq \frac{1 + \alpha_0^2}{\alpha_0^2} \left(\eta(t) + \dot{\sigma}(t) \int_{T_0}^t \frac{\eta(s)}{\alpha_0^2} \exp\left(\frac{\sigma(t)}{\alpha_0^2} - \frac{\sigma(s)}{\alpha_0^2}\right) ds \right) - k' |\dot{d}(t)|$$

where η and σ are functions from $[T_0, T]$ to \mathbb{R}_+ defined by the following formulas:

$$\eta(t) = \dot{\sigma}(t) \left(\|x_0\| + \int_{T_0}^t \beta(s) ds \right) + \beta(t) + k' |\dot{d}(t)|,$$

$$\sigma(t) = \exp\left(\int_{T_0}^t \alpha(s) ds\right).$$

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