

Outer limit of subdifferentials and calmness moduli in linear and nonlinear programming

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Dedicated to Prof. R.T. Rockafellar

- 1 Introduction
- 2 Antecedents and main goals
- 3 Outer limits of subdifferentials

1. Introduction

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- $x \in \mathbb{R}^n$ is the vector of *decision variables*, \mathbb{R}^n equipped with an arbitrary norm, $\|\cdot\|$.
- $f_i \in C^1(\mathbb{R}^n)$, $i = 1, \dots, m$.
- $b \equiv (b_i)_{i=1, \dots, m} \in \mathbb{R}^m$ is the *parameter* to be perturbed.
- The *parameter space*, \mathbb{R}^m , is endowed with

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The associated *feasible set mapping* $\mathcal{F} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is given by

$$\mathcal{F}(b) := \{x \in \mathbb{R}^n : f_i(x) \leq b_i, \text{ for all } i = 1, \dots, m\}. \quad (1)$$

Definition

\mathcal{F} is said to be *calm* at $(\bar{b}, \bar{x}) \in \text{gph}\mathcal{F}$ (the graph of \mathcal{F}) if there exist $\kappa \geq 0$ and neighborhoods U of \bar{x} and V of \bar{b} such that

$$d(x, \mathcal{F}(\bar{b})) \leq \kappa d(b, \bar{b}) \quad (2)$$

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The calmness of \mathcal{F} at (\bar{b}, \bar{x}) is equivalent to the *metric subregularity* of \mathcal{F}^{-1} at (\bar{x}, \bar{b}) , i.e. to the existence of $\kappa \geq 0$ and a (possibly smaller) neighborhood U of \bar{x} such that

$$d(x, \mathcal{F}(\bar{b})) \leq \kappa d(\bar{b}, \mathcal{F}^{-1}(x)), \text{ for all } x \in U. \quad (3)$$

Definition

The *calmness modulus* of \mathcal{F} at (\bar{b}, \bar{x}) , $\text{clm}\mathcal{F}(\bar{b}, \bar{x})$, is the infimum of those $\kappa \geq 0$ for which (2) –or (3)– holds (for some associated neighborhoods). If $\text{clm}\mathcal{F}(\bar{b}, \bar{x}) = +\infty$, \mathcal{F} is *not calm* at (\bar{b}, \bar{x}) .

If $g := \max_{1, \dots, m} g_i$, where $g_i(x) = f_i(x) - b_i$, $i = 1, \dots, m$,

$$\begin{aligned}\sigma(b) &= \{f_i(x) \leq b_i, \quad i = 1, \dots, m\} \\ &= \{g_i(x) \leq 0, \quad i = 1, \dots, m\} \equiv \{g(x) \leq 0\}.\end{aligned}$$

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- Given a nominal feasible solution $\bar{x} \in \mathcal{F}(\bar{b})$, the *set of active indices* at \bar{x} is

$$T_{\bar{b}}(\bar{x}) := \{i \in \{1, \dots, m\} : g_i(\bar{x}) = 0\}.$$

- If $T_{\bar{b}}(\bar{x}) = \emptyset$, \bar{x} is a *Slater point* of $\sigma(\bar{b})$, and one trivially has $\text{clm}\mathcal{F}(\bar{b}, \bar{x}) = 0$. So, we assume

$$T_{\bar{b}}(\bar{x}) \neq \emptyset \text{ or, equivalently, } g(\bar{x}) = 0.$$

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Theorem 1

Let $(\bar{b}, \bar{x}) \in \text{gph}\mathcal{F}$ such that $g(\bar{x}) = 0$. Then we have:

(i) [FHKO'10]

$$\text{clm}\mathcal{F}(\bar{b}, \bar{x}) \leq \left[d_* \left(0_n, \limsup_{x \rightarrow \bar{x}, g(x) > 0} \partial g(x) \right) \right]^{-1}; \quad (4)$$

(ii) [KNT'10] If, additionally, $f_i, i = 1, \dots, m$, are *convex*, then

$$\text{clm}\mathcal{F}(\bar{b}, \bar{x}) = \left[d_* \left(0_n, \limsup_{x \rightarrow \bar{x}, g(x) > 0} \partial g(x) \right) \right]^{-1}, \quad (5)$$

where d_* stands for the distance in \mathbb{R}^n associated with the dual norm $\|\cdot\|_*$ and ∂g represents the *Clarke subdifferential* of g .

REMARKS

- We have taken into account the well-known relationship between $\text{clm}\mathcal{F}(\bar{b}, \bar{x})$ and the *error bound modulus* of g at \bar{x}

$$\text{clm}\mathcal{F}(\bar{b}, \bar{x}) = [\text{Er } g(\bar{x})]^{-1}. \quad (6)$$

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- With respect to (i), in principle, from [FHKO'10] we have

$$\text{Er } g(\bar{x}) \geq \liminf_{x \rightarrow \bar{x}, g(x) > 0} d_* \left(0_n, \hat{\delta}g(x) \right),$$

$\hat{\delta}$ being the *Fréchet subdifferential*. We replace $\hat{\delta}$ by ∂ as consequence of the Clarke regularity of g .

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- With respect to (ii), equality (5) is held under convexity, even **without differentiability assumptions** on the f_i 's (see again [KNT'10, Theorem 1]), in which case, ∂g stands for the usual **subdifferential of convex analysis**.

2. Antecedents and main goals

- For **linear systems**, $\sigma(b) = \{a'_i x \leq b_i, \text{ for all } i = 1, \dots, m\}$, the computation of $\text{clm}\mathcal{F}(\bar{b}, \bar{x})$ is dealt in **[CLPT'14]**. Here, we tackle the case of \mathcal{C}^1 (sometimes convex) - systems.

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- According to Theorem 1, the **key ingredient** is the *outer limit of subdifferentials* of the max-function g , by approaching \bar{x} from outside the feasible set.
- Indeed, the issue of computing the outer limit **has its own interest**. We approached it in **two stages**: firstly, in the **linear case** and, in a second step, in context of \mathcal{C}^1 -systems.
- These results are applied to derive an **upper bound** of $\text{clm}\mathcal{F}(\bar{b}, \bar{x})$ for the \mathcal{C}^1 -system $\sigma(\bar{b})$, and a **lower bound** when the functions f_i , $i = 1, 2, \dots, m$, are **convex**.

For the **linear system** $\sigma(b) = \{a'_i x \leq b_i, \text{ for all } i = 1, \dots, m\}$, we define

$$\mathcal{D}_{\bar{b}}(\bar{x}) := \left\{ \begin{array}{l} D \subset T_{\bar{b}}(\bar{x}) \mid \exists d \text{ s.t. } a'_i d = 1, i \in D, \\ \text{and } a'_i d < 1, i \in T_{\bar{b}}(\bar{x}) \setminus D \end{array} \right\}. \quad (7)$$

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Equivalently, $D \in \mathcal{D}_{\bar{b}}(\bar{x})$ if there exists a **hyperplane** containing $\{a_i, i \in D\}$ such that $\{0_n\} \cup \{a_i, i \in T_{\bar{b}}(\bar{x}) \setminus D\}$ lies on one of the open half-spaces determined by this hyperplane.

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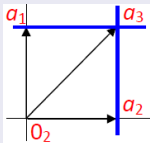
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Example 1

Consider the system (in \mathbb{R}^2 endowed with the Euclidean norm)

$$\{x_2 \leq b_2, x_1 \leq b_1, x_1 + x_2 \leq b_3\}.$$



Take $\bar{b} = 0_3, \bar{x} = 0_2. T_{\bar{b}}(\bar{x}) = \{1, 2, 3\}.$

$$\mathcal{D}_{\bar{b}}(\bar{x}) = \{\{1\}, \{2\}, \{3\}, \{1, 3\}, \{2, 3\}\}.$$

The following theorem, provides a motivation for Theorems 3 and 4 below.

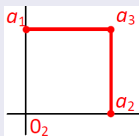
Theorem 2

[CLPT'14, Theorem 4] Given $(\bar{b}, \bar{x}) \in \text{gph}\mathcal{F}$, we have

$$\text{clm}\mathcal{F}(\bar{b}, \bar{x}) = \left(\min_{D \in \mathcal{D}_{\bar{b}}(\bar{x})} d_*(0_n, \text{conv}\{a_i, i \in D\}) \right)^{-1}.$$

Example 1 (revisited)

Recall the previous example: $\{x_2 \leq b_2, x_1 \leq b_1, x_1 + x_2 \leq b_3\}$.



$$d_*(0_2, \text{conv}\{a_1, a_3\}) = d_*(0_2, \text{conv}\{a_2, a_3\}) = 1.$$

$$\text{clm}\mathcal{F}(\bar{b}, \bar{x}) = 1.$$

3. Outer limits of subdifferentials

We are considering the max-function $g : \mathbb{R}^n \rightarrow \mathbb{R}$

$$g(x) := \max_{i=1,\dots,m} g_i(x) \equiv \max_{i=1,\dots,m} \{f_i(x) - b_i\},$$

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As a consequence, g is a *regular function in the sense of Clarke*, and we have

$$\partial g(x) = \text{conv}\{\nabla g_i(x) : i \in I(x)\}$$

(e.g., [Cl'83, 2.3.12]), where

$$I(x) := \{i = 1, \dots, m : g_i(x) = g(x)\}.$$

Now we define

$$\mathcal{D}(x) := \left\{ \begin{array}{l} D \subset I(x) \mid \exists d \text{ s.t. } \nabla g_i(x)' d = 1, i \in D, \\ \text{and } \nabla g_i(x)' d < 1, i \in T_{\bar{b}}(\bar{x}) \setminus D \end{array} \right\}.$$

When $g_i(x) := a_i'x - b_i$, $i = 1, \dots, m$, the corresponding max-function, $g(x) = \max_{i=1, \dots, m} \{a_i'x - b_i\}$, is *polyhedral* and its subdifferential (in the sense of Clarke, which coincides with the usual subdifferential of convex analysis) is

$$\partial g(x) = \text{conv} \{a_i, i \in I(x)\}. \quad (8)$$

Theorem 3

For $g = \max_{i=1, \dots, m} \{a_i'x - b_i\}$, one has

$$\limsup_{x \rightarrow \bar{x}, g(x) > g(\bar{x})} \partial g(x) = \bigcup_{D \in \mathcal{D}(\bar{x})} \text{conv} \{a_i, i \in D\}.$$

Remark

For each $D \in \mathcal{D}(\bar{x})$, $\text{conv}\{a_i, i \in D\}$ is contained in a supporting hyperplane to $\text{conv}\{a_i, i \in I(\bar{x})\} = \partial g(\bar{x})$; then

$$\begin{aligned} \limsup_{x \rightarrow \bar{x}, g(x) > g(\bar{x})} \partial g(x) &= \bigcup_{D \in \mathcal{D}(\bar{x})} \text{conv}\{a_i, i \in D\} \\ &\subset \text{bd conv}\{a_i, i \in I(\bar{x})\} = \text{bd}\partial g(\bar{x}). \end{aligned}$$

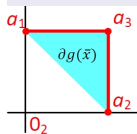
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Example 1 (Revisited; the previous inclusion may be strict.)

Associated with the system of Example 1 (and taking $\bar{b} = 0_3$),
 $g(x) = \max\{x_2, x_1, x_1 + x_2\}$. For $\bar{x} = 0_3$, $I(\bar{x}) = \{1, 2, 3\}$.



$$\begin{aligned} \limsup_{x \rightarrow \bar{x}, g(x) > g(\bar{x})} \partial g(x) &= \text{conv}\{a_1, a_3\} \cup \text{conv}\{a_2, a_3\} \\ \text{bd} \partial g(\bar{x}) &= \text{conv}\{a_1, a_2\} \cup \text{conv}\{a_1, a_3\} \cup \text{conv}\{a_2, a_3\}. \end{aligned}$$

The nonlinear differentiable case

Definition

Associated with $\bar{x} \in \mathbb{R}^n$, we consider the family

$$\mathcal{D}_{AI}(\bar{x}) := \{D \in \mathcal{D}(\bar{x}) : \{\nabla g_i(\bar{x}), i \in D\} \text{ is aff. independent}\}.$$

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Theorem 4

Let $g := \max_{i=1, \dots, m} g_i$, with $g_i \in \mathcal{C}^1(\mathbb{R}^n)$, and $\bar{x} \in \mathbb{R}^n$. Then:

- (i) $\bigcup_{D \in \mathcal{D}_{AI}(\bar{x})} \text{conv} \{\nabla g_i(\bar{x}), i \in D\} \subset \limsup_{x \rightarrow \bar{x}, g(x) > g(\bar{x})} \partial g(x)$;
- (ii) $\limsup_{x \rightarrow \bar{x}, x \neq \bar{x}} \partial g(x) \subset \text{bd } \partial g(\bar{x})$.

Equality in (ii) holds if for all supporting hyperplane H to $\partial g(\bar{x})$, $\{\nabla g_i(\bar{x}), i \in I(\bar{x})\} \cap H$ is affinely independent.

Remark [CHPT'14, Theorem 3.1] shows that equality in (ii) also holds when g is convex, without differentiability assumptions.

Corollary (Estimating the calmness modulus of \mathcal{C}^1 - systems)

Let us consider the \mathcal{C}^1 - system

$$\sigma(b) := \{f_i(x) \leq b_i, \text{ for all } i = 1, \dots, m\},$$

with associated max-function

$$g(x) := \max_{i=1, \dots, m} \{f_i(x) - b_i\}.$$

If $(\bar{b}, \bar{x}) \in \text{gph} \mathcal{F}$ is such that $g(\bar{x}) = 0$, then:

(i) We have

$$\text{clm} \mathcal{F}(\bar{b}, \bar{x}) \leq (d_*(0_n, \text{bd } \partial g(\bar{x})))^{-1}.$$

(ii) If, additionally, functions f_i , $i = 1, \dots, m$, are convex, then

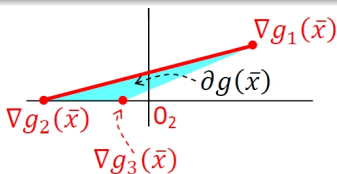
$$\left(\min_{D \in \mathcal{D}_{AI}(\bar{x})} d_*(0_n, \text{conv} \{\nabla f_i(\bar{x}), i \in D\}) \right)^{-1} \leq \text{clm} \mathcal{F}(\bar{b}, \bar{x}).$$

In the following example, the **first inequality** in the previous corollary is **strictly satisfied**, while the **second** is an **equality**.

Example 2

Consider the following system, and take $\bar{x} = 0_2$ and $\bar{b} = 0_3$:

$$\sigma(b) := \left\{ \begin{array}{l} 2x_1^2 + x_2^2 + 4x_1 + 2x_2 \leq b_1, \\ x_1^2 + x_2^2 - 4x_1 \leq b_2, \\ -x_1 \leq b_3, \end{array} \right\}$$



By Theorem 4 (i) :

$$\begin{aligned} \bigcup_{D \in \mathcal{D}_{AI}(\bar{x})} \text{conv} \{ \nabla g_i(\bar{x}), i \in D \} \\ = \text{conv} \{ (-4, 0), (4, 2) \} \end{aligned}$$

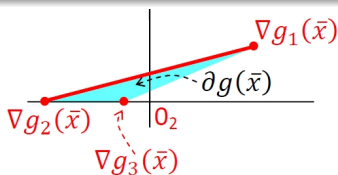
$$\subset \limsup_{x \rightarrow \bar{x}, g(x) > 0} \partial g(x).$$

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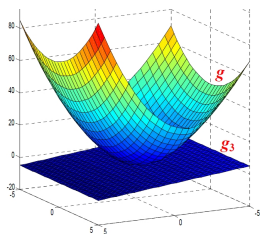
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Let us see that $\limsup_{x \rightarrow \bar{x}, g(x) > 0} \partial g(x) = \text{conv} \{ (-4, 0), (4, 2) \}$.

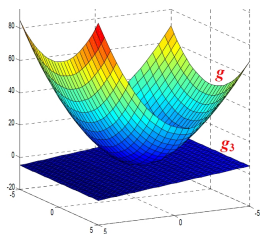


It can be observed that

$$g(x) > 0 \Rightarrow g(x) = \max\{g_1(x), g_2(x)\} > g_3(x).$$

As consequence:

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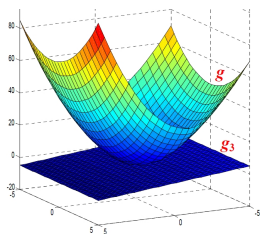
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Then, by Theorem 1(ii),

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$$\begin{aligned} \limsup_{x \rightarrow \bar{x}, g(x) > 0} \partial g(x) &\subset \lim_{x \rightarrow \bar{x}} \text{conv} \{ \nabla g_1(x), \nabla g_2(x) \} \\ &= \text{conv} \{ (-4, 0), (4, 2) \}. \end{aligned}$$

Then, by Theorem 1(ii),

$$\text{clm}\mathcal{F}(\bar{b}, \bar{x}) = [d_* (0_n, \text{conv} \{ (-4, 0), (4, 2) \})]^{-1}.$$

However,

$$\text{bd } \partial g(\bar{x}) = \text{bd } \text{conv} \{ (-4, 0), (4, 2), (-1, 0) \},$$

and then, the inequality (i) in Corollary 1 is strict:

$$\text{clm}\mathcal{F}(\bar{b}, \bar{x}) < [d_* (0_n, \text{bd } \text{conv} \{ (-4, 0), (4, 2), (-1, 0) \})]^{-1}.$$

Optimal set mapping

- Consider the *optimal set mapping* $\mathcal{S} : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ for the LP problem under canonical perturbations:

$$\mathcal{S}(c, b) := \arg \min \{c'x : x \in \mathcal{F}(b)\}. \quad (9)$$

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- Associated with $((\bar{c}, \bar{b}), \bar{x}) \in \text{gph} \mathcal{S}$, we introduce the following family of index subsets associated with the *KKT conditions*:

$$\mathcal{M}_{\bar{c}, \bar{b}}(\bar{x}) := \left\{ \begin{array}{l} D \subset T_{\bar{b}}(\bar{x}) : -\bar{c} \in \text{cone} \{a_i, i \in D\} \\ \text{and } D \text{ is minimal for the inclusion order} \end{array} \right\}.$$

If $D \in \mathcal{M}_{\bar{c}, \bar{b}}(\bar{x})$, Carathéodory's Theorem yields $|D| \leq n$.

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- Associated with each $D \in \mathcal{M}_{\bar{c}, \bar{b}}(\bar{x})$ we consider the mapping $\mathcal{L}_D : \mathbb{R}^m \times \mathbb{R}^D \rightrightarrows \mathbb{R}^n$

$$\mathcal{L}_D(b, d) := \{x \in \mathbb{R}^n : a_i'x \leq b_i, i = 1, \dots, m; -a_i'x \leq d_i, i \in D\}. \quad (10)$$

Theorem

For any $((\bar{c}, \bar{b}), \bar{x}) \in \text{gph}\mathcal{S}$ we have the exact formula





$$\text{clm}\mathcal{S}((\bar{c}, \bar{b}), \bar{x}) = \sup_{D \in \mathcal{M}_{\bar{c}, \bar{b}}(\bar{x})} \text{clm}\mathcal{L}_D((\bar{b}, -\bar{b}_D), \bar{x}), \quad (11)$$





where

$$\mathcal{L}_D(\bar{b}, -\bar{b}_D) = \left\{ x \in \mathbb{R}^n : a'_i x \leq \bar{b}_i, i = 1, \dots, m; -a'_i x \leq -\bar{b}_i, i \in D \right\}.$$

Remark

- 1) \mathcal{L}_D is nothing else but a feasible set mapping (associated to a certain enlarged system).
- 2) The supremum in the right hand side of (11) still constitutes a lower bound for $\text{clm}\mathcal{S}((\bar{c}, \bar{b}), \bar{x})$ in the semi-infinite case.

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