Outer limit of subdifferentials and calmness moduli in linear and nonlinear programming

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Dedicated to Prof. R.T. Rockafellar

1. Introduction

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- $x \in \mathbb{R}^n$ is the vector of *decision variables*, $\mathbb{R}^n$ equipped with an arbitrary norm, $\|\cdot\|$.
- $f_i \in \mathcal{C}^1(\mathbb{R}^n)$, $i = 1, ..., m$.
- $b \equiv (b_i)_{i=1,...m} \in \mathbb{R}^m$ is the *parameter* to be perturbed.
- The *parameter space*, $\mathbb{R}^m$, is endowed with

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The associated *feasible set mapping* $\mathcal{F} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is given by

$$\mathcal{F}(b) := \{x \in \mathbb{R}^n : \ f_i(x) \leq b_i, \text{ for all } i = 1, ..., m\}. \qquad (1)$$

Definition

$\mathcal{F}$ is said to be *calm* at $(\overline{b},\overline{x}) \in \mathrm{gph}\mathcal{F}$ (the graph of $\mathcal{F}$) if there exist $\kappa \geq 0$ and neighborhoods U of $\overline{x}$ and V of $\overline{b}$ such that

$$d(x, \mathcal{F}(\overline{b})) \leq \kappa d(b, \overline{b}) \tag{2}$$

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The calmness of $\mathcal{F}$ at $(\bar{b}, \bar{x})$ is equivalent to the *metric subregularity* of $\mathcal{F}^{-1}$ at $(\bar{x}, \bar{b})$, i.e. to the existence of $\kappa \geq 0$ and a (possibly smaller) neighborhood U of $\bar{x}$ such that

$$d(x, \mathcal{F}(\bar{b})) \leq \kappa d(\bar{b}, \mathcal{F}^{-1}(x)), \text{ for all } x \in U. \tag{3}$$

Definition

The *calmness modulus* of $\mathcal{F}$ at $(\bar{b}, \bar{x})$, $\mathrm{clm}\mathcal{F}(\bar{b}, \bar{x})$, is the infimum of those $\kappa \geq 0$ for which (2) –or (3)– holds (for some associated neighborhoods). If $\mathrm{clm}\mathcal{F}(\bar{b}, \bar{x}) = +\infty$, $\mathcal{F}$ is *not calm* at $(\bar{b}, \bar{x})$.

If $g := \max_{1,\ldots,m} g_i$, where $g_i(x) = f_i(x) - b_i$, $i = 1, \ldots, m$,

$$\begin{aligned} \sigma(b) &= \{f_i(x) \leq b_i,\ \ i = 1, \ldots, m\} \\ &= \{g_i(x) \leq 0,\ \ i = 1, \ldots, m\} \equiv \{g(x) \leq 0\}. \end{aligned}$$

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- Given a nominal feasible solution $\overline{x} \in \mathcal{F}(\overline{b})$, the *set of active indices* at $\overline{x}$ is

$$T_{\overline{b}}(\overline{x}) := \{i \in \{1, \ldots, m\} : \ \ g_i(\overline{x}) = 0\}.$$

- If $T_{\overline{b}}(\overline{x}) = \emptyset$, $\overline{x}$ is a *Slater point* of $\sigma(\overline{b})$, and one trivially has $\mathrm{clm}\mathcal{F}(\overline{b}, \overline{x}) = 0$. So, we assume

$$T_{\overline{b}}(\overline{x}) \neq \emptyset \ \text{ or, equivalently, } g(\overline{x}) = 0.$$

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Theorem 1

Let $(\bar{b},\bar{x}) \in \mathrm{gph}\mathcal{F}$ *such that* $g(\bar{x}) = 0$. *Then we have:*
(i) *[FHKO'10]*

$$\mathrm{clm}\mathcal{F}(\bar{b},\bar{x}) \leq \left[d_* \left(0_n, \limsup_{x \to \bar{x},\, g(x)>0} \partial g(x) \right) \right]^{-1}; \tag{4}$$

(ii) *[KNT'10] If, additionally,* f_i, $i = 1, \ldots, m$, *are convex, then*

$$\mathrm{clm}\mathcal{F}(\bar{b},\bar{x}) = \left[d_* \left(0_n, \limsup_{x \to \bar{x},\, g(x)>0} \partial g(x) \right) \right]^{-1}, \tag{5}$$

where d_* *stands for the distance in* $\mathbb{R}^n$ *associated with the dual norm* $\|\cdot\|_*$ *and* ∂g *represents the Clarke subdifferential of* g.

REMARKS

- We have taken into account the well-known relationship between $\operatorname{clm}\mathcal{F}(\bar{b},\bar{x})$ and the *error bound modulus* of g at $\bar{x}$

$$\operatorname{clm}\mathcal{F}(\bar{b},\bar{x}) = [\operatorname{Er} g(\bar{x})]^{-1}. \tag{6}$$

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- With respect to (i), in principle, from *[FHKO'10]* we have

$$\mathrm{Er}\, g(\bar{x}) \geq \liminf_{x \to \bar{x}, g(x) > 0} d_* \left(0_n, \hat{\partial} g(x)\right),$$

$\hat{\partial}$ being the *Fréchet subdifferential*. We replace $\hat{\partial}$ by ∂ as consequence of the Clarke regularity of g.

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  $\hat{\partial}$ being the *Fréchet subdifferential*. We replace $\hat{\partial}$ by ∂ as consequence of the Clarke regularity of g.
- With respect to *(ii)* , equality (5) is held under convexity, even without differentiability assumptions on the f_i's (see again [KNT'10, Theorem 1]), in which case, ∂g stands for the usual subdifferential of convex analysis.

2. Antecedents and main goals

- For linear systems, $\sigma(b) = \{a_i' x \leq b_i, \text{ for all } i = 1, ..., m\}$, the computation of $\text{clm}\mathcal{F}(\bar{b}, \bar{x})$ is dealt in [CLPT'14] . Here, we tackle the case of $\mathcal{C}^1$ (sometimes convex) - systems.

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- According to Theorem 1, the key ingredient is the *outer limit of subdifferentials* of the max-function g, by approaching $\overline{x}$ from outside the feasible set.
- Indeed, the issue of computing the outer limit has its own interest . We approached it in two stages: firstly, in the linear case and, in a second step, in context of $\mathcal{C}^1$-systems.
- These results are applied to derive an *upper bound* of $\mathrm{clm}\mathcal{F}(\overline{b}, \overline{x})$ for the $\mathcal{C}^1$-system $\sigma(\overline{b})$, and a *lower bound* when the functions f_i, $i = 1, 2, ..., m$, are convex.

For the linear system $\sigma(b) = \{a_i' x \leq b_i, \text{ for all } i = 1, ..., m\}$, we define

$$\mathcal{D}_{\overline{b}}(\overline{x}) := \left\{ \begin{array}{c} D \subset T_{\overline{b}}(\overline{x}) \mid \exists d \text{ s.t. } a_i' d = 1,\ i \in D, \\ \text{and } a_i' d < 1,\ i \in T_{\overline{b}}(\overline{x}) \setminus D \end{array} \right\}. \tag{7}$$

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Equivalently, $D \in \mathcal{D}_{\overline{b}}(\overline{x})$ if there exists a **hyperplane** containing $\{a_i, i \in D\}$ such that $\{0_n\} \cup \{a_i, i \in T_{\overline{b}}(\overline{x}) \setminus D\}$ lies on one of the open half-spaces determined by this hyperplane.

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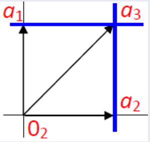
Example 1

Consider the system (in $\mathbb{R}^2$ endowed with the Euclidean norm)

$$\{x_2 \leq b_2,\ x_1 \leq b_1,\ x_1 + x_2 \leq b_3\}.$$



Take $\overline{b} = 0_3$, $\overline{x} = 0_2$. $T_{\overline{b}}(\overline{x}) = \{1, 2, 3\}$.

$\mathcal{D}_{\overline{b}}(\overline{x}) = \{\{1\}, \{2\}, \{3\}, \{1, 3\}, \{2, 3\}\}$.

The following theorem, provides a motivation for Theorems 3 and 4 below.

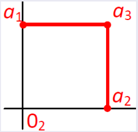
Theorem 2

[CLPT'14, Theorem 4] Given $(\bar{b},\bar{x}) \in \mathrm{gph}\mathcal{F}$, *we have*

$$\mathrm{clm}\mathcal{F}(\bar{b},\bar{x}) = \left(\min_{D \in \mathcal{D}_{\bar{b}}(\bar{x})} d_* \left(0_n, \mathrm{conv}\left\{a_i,\ i \in D\right\}\right)\right)^{-1}.$$

Example 1 (revisited)

Recall the previous example: $\{\ x_2 \leq b_2,\ x_1 \leq b_1,\ x_1 + x_2 \leq b_3\}$.



$$d_*(0_2, \mathrm{conv}\{a_1, a_3\}) = d_*(0_2, \mathrm{conv}\{a_2, a_3\}) = 1.$$

$$\mathrm{clm}\mathcal{F}(\bar{b},\bar{x}) = 1.$$

3. Outer limits of subdiferentials

We are considering the max-function $g : \mathbb{R}^n \to \mathbb{R}$

$$g(x) := \max_{i=1,\dots,m} g_i(x) \equiv \max_{i=1,\dots,m} \{f_i(x) - b_i\},$$

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As a consequence, g is a *regular function in the sense of Clarke*, and we have

$$\partial g(x) = \operatorname{conv}\{\nabla g_i(x) : i \in I(x)\}$$

(e.g., [Cl'83, 2.3.12]), where

$$I(x) := \{i = 1, \dots, m : g_i(x) = g(x)\}.$$

Now we define

$$\mathcal{D}(x) := \left\{ \begin{array}{c} D \subset I(x) \mid \exists d \text{ s.t. } \nabla g_i(x)' d = 1,\ i \in D, \\ \text{and } \nabla g_i(x)' d < 1,\ i \in T_{\overline{b}}(\overline{x}) \setminus D \end{array} \right\}.$$

When $g_i(x) := a_i' x - b_i$, $i = 1, ..., m$, the corresponding max-function, $g(x) = \max_{i=1,...,m}\{a_i' x - b_i\}$, is *polyhedral* and its subdifferential (in the sense of Clarke, which coincides with the usual subdifferential of convex analysis) is

$$\partial g(x) = \operatorname{conv}\{a_i,\ i \in I(x)\}. \tag{8}$$

Theorem 3

For $g = \max\limits_{i=1,...,m}\{a_i' x - b_i\}$, *one has*

$$\limsup_{x \to \overline{x},\ g(x) > g(\overline{x})} \partial g(x) = \bigcup_{D \in \mathcal{D}(\overline{x})} \operatorname{conv}\{a_i,\ i \in D\}.$$

Remark

For each $D \in \mathcal{D}(\overline{x})$, $\operatorname{conv}\{a_i, i \in D\}$ *is contained in a supporting hyperplane to* $\operatorname{conv}\{a_i,\ i \in I(\overline{x})\} = \partial g(\overline{x})$*; then*

$$\begin{aligned} \limsup_{x \to \overline{x},\ g(x) > g(\overline{x})} \partial g(x) &= \bigcup_{D \in \mathcal{D}(\overline{x})} \operatorname{conv}\{a_i,\ i \in D\} \\ &\subset \operatorname{bd}\operatorname{conv}\{a_i,\ i \in I(\overline{x})\}) = \operatorname{bd}\partial g(\overline{x}). \end{aligned}$$

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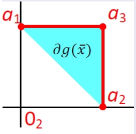
Example 1 (Revisited; the previous inclusion may be strict.)

Associated with the system of Example 1 (and taking $\overline{b} = 0_3$*),*
$g(x) = \max\{x_2, x_1, x_1 + x_2\}$. *For* $\overline{x} = 0_3, I(\overline{x}) = \{1, 2, 3\}$.



$$\limsup_{x \to \overline{x},\ g(x) > g(\overline{x})} \partial g(x) = \operatorname{conv}\{a_1, a_3\} \cup \operatorname{conv}\{a_2, a_3\}$$

$$\operatorname{bd} \partial g(\overline{x}) = \operatorname{conv}\{a_1, a_2\} \cup \operatorname{conv}\{a_1, a_3\} \cup \operatorname{conv}\{a_2, a_3\}.$$

The nonlinear differentiable case

Definition

Associated with $\overline{x} \in \mathbb{R}^n$, we consider the family

$$\mathcal{D}_{AI}(\overline{x}) := \{D \in \mathcal{D}(\overline{x}) : \{\nabla g_i(\overline{x}), i \in D\} \text{ is aff. independent}\}.$$

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Theorem 4

Let $g := \max_{i=1,\ldots,m} g_i$, *with* $g_i \in \mathcal{C}^1(\mathbb{R}^n)$, *and* $\overline{x} \in \mathbb{R}^n$. *Then:*

(i) $\bigcup_{D \in \mathcal{D}_{AI}(\overline{x})} \operatorname{conv}\{\nabla g_i(\overline{x}), i \in D\} \subset \limsup_{x \to \overline{x},\, g(x) > g(\overline{x})} \partial g(x)$;

(ii) $\limsup_{x \to \overline{x},\, x \neq \overline{x}} \partial g(x) \subset \operatorname{bd} \partial g(\overline{x})$.

Equality in (ii) *holds if for all supporting hyperplane* H *to* $\partial g(\bar{x})$, $\{\nabla g_i(\bar{x}), i \in I(\bar{x})\} \cap H$ *is affinely independent.*

Remark *[CHPT'14, Theorem 3.1] shows that equality in* (ii) *also holds when* g *is convex, without differentiability assumptions.*

Corollary (Estimating the calmness modulus of $\mathcal{C}^1$- systems)

Let us consider the $\mathcal{C}^1$- system

$$\sigma(b) := \{f_i(x) \leq b_i, \text{ for all } i = 1, ..., m\},$$

with associated max-function

$$g(x) := \max_{i=1,...,m} \{f_i(x) - b_i\}.$$

If $(\bar{b}, \bar{x}) \in \mathrm{gph}\mathcal{F}$ is such that $g(\bar{x}) = 0$, then:

(i) We have

$$\mathrm{clm}\mathcal{F}(\bar{b}, \bar{x}) \leq (d_*(0_n, \mathrm{bd}\, \partial g(\bar{x})))^{-1}.$$

(ii) If, additionally, functions f_i, $i = 1, ..., m$, are convex, then

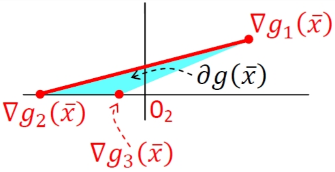
$$\left(\min_{D \in \mathcal{D}_{AI}(\bar{x})} d_*(0_n, \mathrm{conv}\{\nabla f_i(\bar{x}),\ i \in D\})\right)^{-1} \leq \mathrm{clm}\mathcal{F}(\bar{b}, \bar{x}).$$

In the following example, the first inequality in the previous corollary is strictly satisfied , while the second is an equality.

Example 2

Consider the following system, and take $\overline{x} = 0_2$ *and* $\overline{b} = 0_3$:

$$\sigma(b) := \left\{ \begin{array}{l} 2x_1^2 + x_2^2 + 4x_1 + 2x_2 \leq b_1, \\ x_1^2 + x_2^2 - 4x_1 \leq b_2, \\ -x_1 \leq b_3, \end{array} \right\}$$



By Theorem 4 (i) :

$$\bigcup_{D\in\mathcal{D}_{AI}(\bar{x})} \text{conv}\{\nabla g_i(\bar{x}), i \in D\}$$
$$= \text{conv}\{(-4,0),(4,2)\}$$
$$\subset \limsup_{x\to\bar{x},\, g(x)>0} \partial g(x).$$

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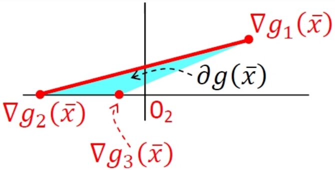

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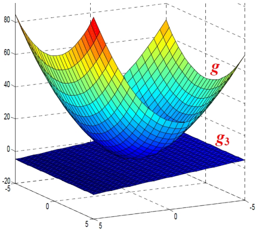
Let us see that $\limsup_{x \to \bar{x},\ g(x)>0} \partial g(x) = \text{conv}\left\{(-4,0),(4,2)\right\}$.



It can be observed that

$$g(x) > 0 \Rightarrow g(x) = \max\{g_1(x), g_2(x)\} > g_3(x).$$

As consequence:

$$\limsup_{x \to \bar{x},\, g(x) > 0} \partial g(x) \subset \lim_{x \to \bar{x}} \operatorname{conv} \{\nabla g_1(x), \nabla g_2(x)\}\}$$

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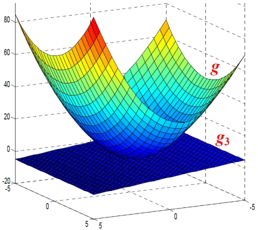

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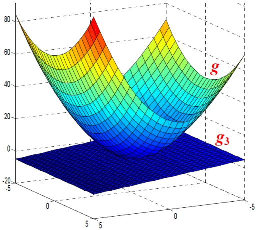
Then, by Theorem 1(ii),

$$\operatorname{clm}\mathcal{F}(\bar{b},\bar{x}) = \left[d_*\left(0_n, \operatorname{conv}\{(-4,0),(4,2)\}\right)\right]^{-1}.$$



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$$= \operatorname{conv}\{(-4,0),(4,2)\}.$$

Then, by Theorem 1(ii),

$$\operatorname{clm}\mathcal{F}(\bar{b}, \bar{x}) = [d_*(0_n, \operatorname{conv}\{(-4,0),(4,2)\})]^{-1}.$$

However,

$$\operatorname{bd} \partial g(\bar{x}) = \operatorname{bd} \operatorname{conv}\{(-4,0),(4,2),(-1,0)\},$$

and then, the inequality (i) in Corollary 1 is strict:

$$\operatorname{clm}\mathcal{F}(\bar{b}, \bar{x}) < [d_*(0_n, \operatorname{bd}\operatorname{conv}\{(-4,0),(4,2),(-1,0)\})]^{-1}.$$

Optimal set mapping

- Consider the *optimal set mapping* $\mathcal{S} : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ for the *LP problem* under *canonical perturbations*:

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- Associated with $\left((\bar{c}, \bar{b}), \bar{x}\right) \in \operatorname{gph}\mathcal{S}$, we introduce the following family of index subsets associated with the *KKT conditions*:

$$\mathcal{M}_{\bar{c},\bar{b}}(\bar{x}) := \left\{ \begin{array}{c} D \subset T_{\bar{b}}(\bar{x}) : \ -\bar{c} \in \text{cone}\{a_i, \ i \in D\} \\ \text{and } D \text{ is minimal for the inclusion order} \end{array} \right\}.$$

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  If $D \in \mathcal{M}_{\bar{c},\bar{b}}\left(\bar{x}\right)$, Carathéodory's Theorem yields $|D| \leq n$.

- Associated with each $D \in \mathcal{M}_{\bar{c},\bar{b}}\left(\bar{x}\right)$ we consider the mapping $\mathcal{L}_D : \mathbb{R}^m \times \mathbb{R}^D \rightrightarrows \mathbb{R}^n$

$$\mathcal{L}_D\left(b,d\right) := \left\{x \in \mathbb{R}^n : \ a_i'x \leq b_i, \ i = 1,..,m; \ -a_i'x \leq d_i, \ i \in D\right\}. \tag{10}$$

Theorem

For any $\left((\bar{c},\bar{b}),\bar{x}\right) \in \mathrm{gph}\mathcal{S}$ *we have the exact formula*

$$\mathrm{clm}\mathcal{S}\left((\bar{c},\bar{b}),\bar{x}\right) = \sup_{D\in\mathcal{M}_{\bar{c},\bar{b}}(\bar{x})} \mathrm{clm}\mathcal{L}_D\left(\left(\bar{b},-\bar{b}_D\right),\bar{x}\right), \qquad (11)$$

where

$$\mathcal{L}_D\left(\bar{b},-\bar{b}_D\right) = \left\{x\in\mathbb{R}^n:\ a_i'x\leq\bar{b}_i,\ i=1,..,m;\ -a_i'x\leq-\bar{b}_i,\ i\in D\right\}.$$

Remark

1) $\mathcal{L}_D$ *is nothing else but a feasible set mapping (associated to a certain enlarged system).*
2) The supremum in the right hand side of (11) still constitutes a lower bound for $\mathrm{clm}\mathcal{S}\left((\bar{c},\bar{b}),\bar{x}\right)$ *in the semi-infinite case.*

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