FULL STABILITY OF VARIATIONAL SYSTEMS

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PARAMETRIC VARIATIONAL SYSTEMS (PVS)

given in the following form

$$v \in f(x,p) + \partial_x g(x,p)$$

where $x \in X$ is the decision variable from a Hilbert space X, $v \in X$ signifies canonical perturbations while $p \in P$ basic ones from a metric space P, f is smooth around $(\bar{x}, \bar{p})$, g is extended-real-valued l.s.c., ∂_x stands for the partial limiting subdifferential

Solution map to (PVS) is

$$S(v,p) := \left\{ x \in X \,\middle|\, v \in f(x,p) + \partial_x g(x,p) \right\}$$

FULL STABILITY OF LOCAL MINIMIZERS

We say that $\bar{x}$ is a Lipschitzian fully stable local minimizer of $g\colon X \times P \to \overline{I\!R}$ relative to $\bar{p} \in P$ if there exist $\kappa, \ell, \gamma > 0$ and a ngbh $V \times Q$ of $(\bar{v}, \bar{p})$ such that the argminimum mapping

$$(v,p) \mapsto M_\gamma(v,p) := \text{argmin}\Big\{g(x,p) - \langle v, x\rangle \Big|\ \|x - \bar{x}\| \le \gamma\Big\}$$

is single-valued on $V \times Q$ with $M_\gamma(\bar{v}, \bar{p}) = \bar{x}$, satisfying

$$\|M_\gamma(v_1,p_1) - M_\gamma(v_2,p_2)\| \le \kappa\|v_1 - v_2\| + \ell d(p_1,p_2)$$

and in addition the local value function

$$(v,p) \mapsto m_\gamma(v,p) := \text{inf}\Big\{g(x,p) - \Big\langle v, x\Big\rangle \Big|\ \|x - \bar{x}\| \le \gamma\Big\}$$

is also locally Lipschitz continuous around $(\bar{v}, \bar{p})$.

The Hölder full stability postulates

$$\|M_\gamma(v_1,p_1) - M_\gamma(v_2,p_2)\| \le \kappa\|v_1 - v_2\| + \ell d(p_1,p_2)^{\frac{1}{2}}$$

LOCAL MONOTONICITY

DEFINITION Let $T\colon X \rightrightarrows X$ be a set-valued operator in a Hilbert space, and let $(\bar{x}, \bar{v}) \in \operatorname{gph} T$. We say that

(i) T is locally strongly monotone around $(\bar{x}, \bar{v})$ with modulus $\kappa > 0$ if there is a neighborhood $U \times V$ of $(\bar{x}, \bar{v})$ such that

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geq \kappa \|u_1 - u_2\|^2, \quad (u_1, v_1), (u_2, v_2) \in \operatorname{gph} T \cap (U \times V)$$

(ii) T is locally strongly maximal monotone around $(\bar{x}, \bar{v})$ with modulus $\kappa > 0$ if there is a neighborhood $U \times V$ such that the above inequality holds and that $\operatorname{gph} T \cap (U \times V) = \operatorname{gph} S \cap (U \times V)$ for any monotone operator S with $\operatorname{gph} T \cap (U \times V) \subset \operatorname{gph} S$

(iii) T is locally hypomonotone around $(\bar{x}, \bar{v})$ if there is a neighborhood $U \times V$ of this point and $r > 0$ such that

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geq -r\|u_1 - u_2\|^2, \quad (u_1, v_1), (u_2, v_2) \in \text{gph}\, T \cap (U \times V)$$

STRONG MONOTONICITY VIA LOCALIZATION

THEOREM For $T : X \rightrightarrows X$ in Hilbert spaces the following assertions are equivalent

(i) T is locally strongly maximally monotone around $(\bar{x}, \bar{v}) \in \operatorname{gph} T$ with modulus $\kappa > 0$

(ii) T is locally strongly monotone around $(\bar{x}, \bar{v})$ with modulus κ and the inverse mapping T^{-1} admits a Lipschitz continuous single-valued localization around $(\bar{v}, \bar{x})$

(iii) The mapping T^{-1} admits a single-valued localization ϑ relative to a neighborhood $V \times U$ of $(\bar{v}, \bar{x})$ such that for all $v_1, v_2 \in V$ we have the estimate

$$\Big\|(v_1 - v_2) - 2\kappa\Big[\vartheta(v_1) - \vartheta(v_2)\Big]\Big\| \le \|v_1 - v_2\|$$

CODERIVATIVES

Given $T\colon X \rightrightarrows X$ and $(\bar{x},\bar{y}) \in \operatorname{gph} T$, the regular coderivative of T at $(\bar{x},\bar{y})$ is defined by

$$\widehat{D}^*T(\bar{x},\bar{y})(u) := \left\{ v \in X \,\middle|\, \limsup_{\substack{(x,y)\to(\bar{x},\bar{y})\\ y\in T(x)}} \frac{\langle u, x-\bar{x}\rangle - \langle v, y-\bar{y}\rangle}{\|x-\bar{x}\| + \|y-\bar{y}\|} \le 0 \right\}$$

The limiting coderivative of T at $(\bar{x},\bar{y})$ is

$$D^*T(\bar{x},\bar{y})(\bar{u}) := \left\{ \bar{v} \,\middle|\, \exists\,(x_k,y_k,u_k,v_k) \to (\bar{x},\bar{y},\bar{u},\bar{v}),\ v_k \in \widehat{D}^*T(x_k,y_k)(u_k) \right\}$$

The limiting coderivative D^*T enjoys full pointwise calculus

NGBH CHARACT. OF LOCAL STRONG MAX MONOTONICITY

THEOREM Let $T\colon X \rightrightarrows X$ be of closed graph around the point $(\bar{x},\bar{v}) \in \text{gph}\, T$. The following are equivalent

(i) T is locally strongly maximal monotone around $(\bar{x},\bar{v})$ with modulus $\kappa > 0$

(ii) T is locally hypomonotone around $(\bar{x},\bar{v})$ and there is $\eta > 0$ such that

$\langle z, w\rangle \geq \kappa\|w\|^2$ for all $z \in \widehat{D}^*T(u,v)(w),\ (u,v) \in \text{gph}\, T \cap B_\eta(\bar{x},\bar{v})$

The conditions in **(ii)** ensure the strong metric regularity of T around $(\bar{x},\bar{v})$ with modulus κ^{-1}

POINT. CHARACT. OF LOCAL STRONG MAX MONOTON.

THEOREM Let $T\colon I\!R^n \to I\!R^n$ be Lipschitz continuous around $\bar{x}$. The following are equivalent

(i) T is locally strongly monotone around $(\bar{x}, T(\bar{x}))$ with some modulus $\kappa > 0$

(ii) $D^*T(\bar{x})$ is positive-definite in the sense that

$$\langle z, w \rangle > 0 \text{ whenever } z \in D^*T(\bar{x})(w),\ w \neq 0$$

LIPSCHITZ AND HÖLDER FULL STABILITY OF PVS

DEFINITION (i) $\bar{x}$ is a Lipschitzian fully stable solution to PVS for the parameter pair $(\bar{v},\bar{p})$ if the solution map admits a single-valued localization ϑ relative to some neighborhood $V\times Q\times U$ of $(\bar{v},\bar{p},\bar{x})$ such that for any $(v_1,p_1),(v_2,p_2)\in V\times Q$ we have

$$\Big\|(v_1-v_2)-2\kappa\Big[\vartheta(v_1,p_1)-\vartheta(v_2,p_2)\Big]\Big\|\le\|v_1-v_2\|+\ell d(p_1,p_2)$$

with some positive constants κ and ℓ

(ii) $\bar{x}$ is a Hölderian fully stable solution to PVS if

$$\Big\|(v_1-v_2)-2\kappa[\vartheta(v_1,p_1)-\vartheta(v_2,p_2)]\Big\|\le\|v_1-v_2\|+\ell d(p_1,p_2)^{\frac{1}{2}}$$

with some positive constants κ and ℓ

SUBDIFFERENTIALS

for $g\colon X \to (-\infty, \infty]$ with $\bar{x} \in \operatorname{dom} g$, $E_g(x) = \{\alpha \in \mathbb{R} \mid \alpha \geq g(x)\}$

regular

$$\widehat{\partial} g(\bar{x}) = \widehat{D}^* E_g\Big(\bar{x}, g(\bar{x})\Big)(1)$$

limiting

$$\partial g(\bar{x}) = D^* E_g\Big(\bar{x}, g(\bar{x})\Big)(1)$$

singular/horizon

$$\partial^\infty g(\bar{x}) = D^* E_g\Big(\bar{x}, g(\bar{x})\Big)(0)$$

STANDING ASSUMPTIONS FOR FULL STABILITY

(A1) f is smooth in x around $(\bar{x},\bar{p})$ uniformly in p and

$$\|f(x,p_1)-f(x,p_2)\|\le Ld(p_1,p_2),\quad x\in U,\ p_1,p_2\in Q$$

(A2) g is parametrically continuously prox-regular at $(\bar{x},\bar{p})$ for $\widehat{v}=\bar{v}-f(\bar{x},\bar{p})\in\partial_x g(\bar{x},\bar{p})$

(A3) The following basic constraint qualification (BCQ) holds

$$(0,q)\in\partial^\infty g(\bar{x},\bar{p})\Longrightarrow q=0$$

2nd-ORDER CHARACT. OF HÖLDER FULL STABILITY

THEOREM Let (A1)–(A3) hold in Hilbert spaces. Then the following are equivalent

(i) $\bar{x}$ is a Hölderian fully stable solution of PVS corresponding to the parameter pair $(\bar{v}, \bar{p})$ with the moduli $\kappa, \ell > 0$

(ii) There exists a number $\eta > 0$ such that whenever $(u, p, v) \in$ gph $\partial_x g \cap \mathbb{B}_\eta(\bar{x}, \bar{p}, \bar{v})$ we have

$$\langle \nabla_x f(\bar{x}, \bar{p}) w, w \rangle + \langle z, w \rangle \geq \kappa \|w\|^2 \quad \text{for} \quad z \in (\widehat{D}^* \partial g_p)(u, v)(w), \; w \in X$$

where $g_p(x) = g(x, p)$

2nd-ORDER CHARACT. OF LIPSCHITZ FULL STABILITY

THEOREM Under the validity of (A1)–(A3) in finite dimensions the Lipschitzian full stability of $\bar{x} \in S(\bar{v}, \bar{p})$ for PVS with some modulus $\kappa > 0$ is equivalent to the simultaneous validity of the following pointwise conditions

$\langle \nabla_x f(\bar{x}, \bar{p})w, w\rangle + \langle z, w\rangle > 0$ for all $(z, q) \in \Big(D^*\partial_x g\Big)(\bar{x}, \bar{p}, \widehat{v})(w),\ w \neq 0$

$$(0, q) \in \Big(D^*\partial_x g\Big)(\bar{x}, \bar{p}, \widehat{v})(0) \Longrightarrow q = 0$$

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