A few major encounters with Terry

- **1974** Introduced to Terry in my first Ph.D. year in the Department of Operations Research at Stanford University on his visit as a seminar speaker

- **1978** Met Terry's first Ph.D. graduate Lynn McLinden (and Terry) at a conference in Erice Italy: 2 significant events at this meeting

- **2004** Terry was the first plenary speaker at the inaugural triennial ICCOPT held at Rensselaer Polytechnic Institute

- **2005** Was honored to be invited to help celebrate Terry's 70 birthday hosted by Jie Sun

- **2007** Guest of Terry at his resort home in Whidbey Island off Seattle.

From my Ph.D. days till now and extending to the future, Terry's work lays the foundation for all of my work (and of course others too).

Happy 80th Birthday, Terry!

And many more

On Stochastic Nash Equilibrium Problems

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Reporting joint work with Suvrajeet Sen (University of Southern California) and Uday Shanbhag (Penn State University)

Research is built on areas pioneered by Terry:

- Convexity and beyond
- Stochastic programming
- Deviation and risk measures
- Stochastic variational inequalities.

Contents of Presentation

- State of the art of deterministic non-cooperative games
- A class of mean-deviation-composite game
- Some challenging issues
- Inner versus outer iterations: best response and sample approximations
- The case of private recourse

The abstract generalized Nash equilibrium problem

deterministic, one stage, coupled constraints

N selfish players each (labeled $i = 1, \cdots, N$) with

- a moving strategy set $\Xi^i(x^{-i}) \subseteq X^i \subseteq \mathbb{R}^{n_i}$, and
- a cost function $\zeta_i(\bullet, x^{-i}) : \mathbb{R}^{n_i} \to \mathbb{R}$,

both dependent on the rivals' strategy tuple $x^{-i} \triangleq \left(x^{i'}\right)_{i' \neq i} \in \mathbb{R}^{-i} \triangleq \prod_{i' \neq i} \mathbb{R}^{n_{i'}}$.

Anticipating rivals' strategy x^{-i}, player i solves: $\boxed{\underset{x^i \in \Xi^i(x^{-i})}{\text{minimize}} \;\; \zeta_i(x^i, x^{-i})}$

A Nash equilibrium (NE) is a strategy tuple $\mathbf{x}^* \triangleq \left(x^{*,i}\right)_{i=1}^N$ such that

$$x^{*,i} \in \underset{x^i \in \Xi^i(x^{*,-i})}{\text{argmin}} \;\; \zeta_i(x^i, x^{*,-i}), \quad \forall\, i = 1, \cdots, N.$$

In words, no player can improve individual objective by unilaterally deviating from an equilibrium strategy.

The uncoupled case: $\Xi^i(x^{-i}) = X^i$ for all $i = 1, \cdots N$.

State of the art

- $\zeta_i(x)$ is finite valued for all $x \in \mathbf{X} \triangleq \prod_{i=1}^{N} X^i$;

- $\zeta_i(x^i, x^{-i})$ is convex in x^i for fixed x^{-i} (an advanced, albeit limited, treatment of nonconvexity is possible)

- algorithmically, $\zeta_i(\bullet, x^{-i})$ is required to be continuously differentiable for treatment by a variational approach

- the partial gradient $\nabla_{x^i}\zeta_i(x)$ is required to be continuously differentiable, with dominant $\nabla^2_{x^i}\zeta_i(x)$ over $\nabla^2_{x^i x^{i'}}\zeta_i(x)$ for $i' \neq i$, for the convergence of the best-response algorithm

- convergence of the best-response algorithm is so far restricted to the uncoupled case

- other solution approaches exist, e.g., under monotonicity; but the best-response approach is the most effective in a distributed environment where communication among agents is limited.

A class of mean-deviation composite game

$$\zeta_i(x) \triangleq \underbrace{\theta_i(x)}_{\text{deterministic first-stage objective}} + \mathbb{E}\left[\underbrace{\psi_i(x,\widetilde{\omega})}_{\text{second-stage quadratic recourse}}\right] + \underbrace{\lambda_i}_{\text{positive weight}} \underbrace{\mathcal{D}_i\left[\,\psi_i(x,\widetilde{\omega})\,\right]}_{\text{deviation measure}},$$

where $$\psi_i(x;\omega) \triangleq \underset{z^i}{\text{minimum}} \left[f^{\,i}(\omega) + \underbrace{\sum_{j \neq i} G^{\,ij}(\omega) x^j}_{\text{no } x^i} \right]^T z^i + \tfrac{1}{2}\,(\,z^i\,)^T \underbrace{Q^{\,i}}_{\text{spsd}} z^i$$

$$\text{subject to} \quad \sum_{j=1}^{N} C^{\,ij}(\omega) x^j + \underbrace{D^{\,i}}_{\text{simple recourse}} z^i \geq \xi^{\,i}(\omega).$$

Several technical challenges

• Possible infeasibility and/or unboundedness of recourse functions jeopardizes a constructive treatment. [Remedy: relatively complete recourse function]

• Convexity of $\zeta_i(\bullet, x^{-i})$ could be jeopardized by a general deviation measure that has the following representation:

$$\mathcal{D}(\mathcal{Z}) \triangleq \mathbb{E}\mathcal{Z} - \inf_{Q \in \mathcal{Q}} \mathbb{E}\left[\mathcal{Z}Q\right],$$

with $\mathcal{Q}$ being the risk envelope associated with $\mathcal{D}$ (Rockafellar, Uryasev, and Zabarankin 2006) [Remedy: focus on a class of deviations]

• The possible non-uniqueness of the minimizer of the value function $\psi_i(x;\omega)$, jeopardizes the differentiability in x^i [Remedy: regularization]

• The expectation operator needs to be approximated [Remedy: sampling and/or progressive hedging]

• Monotonicity in a resulting variational formulation, if applicable, is highly unlikely [Remedy: best-response]

• Coupled first-stage constraints $\Xi^i(x^i)$ and coupled second-stage constraints in recourse complicate treatment [Remedy: focus on the uncoupled case]

Relatively complete recourse

For all $x \in \mathbf{X}$ and almost all $\omega \in \Omega$,

$$\left[\, (D^{\,i})^T v^i \,=\, 0,\;\; v^i \,\geq\, 0 \,\right] \;\; \overset{====>}{\text{primal feas}} \qquad \left[\, \sum_{j=1}^{N} C^{\,ij}(\omega) x^j - \xi^{\,i}(\omega) \,\right]^T v^i \,\geq\, 0,$$

$$D^{\,i} z^i \,\geq\, 0,\, Q z^i \,=\, 0 \;\; \overset{====>}{\text{primal bddness}} \;\; \left[\, f^{\,i}(\omega) + \sum_{j \neq i} G^{\,ij}(\omega) x^j \,\right]^T z^i \,\geq\, 0.$$

Dual program:

$$\begin{array}{ll} \underset{z^i,\, v^i}{\text{maximize}} & \left[\, \xi^{\,i}(\omega) - \sum_{j=1}^{N} C^{\,ij}(\omega) x^j \,\right]^T v^i - \frac{1}{2}\,(\, z^i \,)^T Q^{\,i} z^i \\ \text{subject to} & -Q^{\,i} z^i + (\, D^{\,i} \,)^T v^i \,=\, f^{\,i}(\omega) + \sum_{j \neq i} G^{\,ij}(\omega) x^j \\ \text{and} & v^i \,\geq\, 0. \end{array}$$

Classes of deviation measures

- quantile or CVAR based: for $\gamma \in [0,1]$, let

$$\begin{aligned}\mathcal{D}^{\mathrm{QD}}_{\gamma}(\mathcal{Z}) &\triangleq \mathbb{E}\left\{\gamma\,[\,\mathcal{Z}-\kappa_\gamma(\mathcal{Z})\,]_+ + (\,1-\gamma\,)\,[\,\mathcal{Z}-\kappa_\gamma(\mathcal{Z})\,]_-\,\right\}\\ &= \underset{t\in\mathbb{R}}{\mathrm{minimum}}\,\mathbb{E}\left[\gamma\,(\,\mathcal{Z}-t\,)_+ + (\,1-\gamma\,)\,(\,\mathcal{Z}-t\,)_-\right],\end{aligned}$$

with the minimizer being the γ-quantile $\kappa_\gamma(\mathcal{Z})$ of the random variable $\mathcal{Z}$.

- absolute semi-deviation (ASD):

$$\begin{aligned}\mathcal{D}^{\mathrm{ASD}}(\mathcal{Z}) &\triangleq \mathbb{E}\,[\mathcal{Z}-\mathbb{E}\mathcal{Z}]_+ = \underset{\gamma\in[0,1]}{\mathrm{maximum}}\,\mathcal{D}^{\mathrm{QD}}_{\gamma}(\mathcal{Z}),\\ &= \max_{\gamma\in[0,1]}\left\{\min_{t\in\mathbb{R}}\mathbb{E}\left[\gamma\,(\,\mathcal{Z}-t\,)_+ + (\,1-\gamma\,)\,(\,\mathcal{Z}-t\,)_-\right]\right\}\end{aligned}$$

proved by Ogryczak and Ruszczyński 2002

- absolute deviation (AD):

$$\mathcal{D}^{\mathrm{AD}}(\mathcal{Z}) \triangleq \mathbb{E}\,|\,\mathcal{Z}-\mathbb{E}\mathcal{Z}\,| = 2\,\mathbb{E}\,[\,\mathcal{Z}-\mathbb{E}\mathcal{Z}\,]_+\,.$$

Want a unified treatment of the above deviation functions.

A unified mean-deviation function

For a parameter $\lambda > 0$, let $H_\lambda(z;t;\gamma)$ be a function of the triple $(z;t;\gamma)$ and consider

$$\widehat{\varphi}_\lambda(x) \triangleq \max_{\gamma\in\Gamma} \min_{t\in\mathbb{R}} \mathbb{E}\left[\, H_\lambda(\psi(x;\widetilde{\omega});t;\gamma)\,\right], \quad \text{where } \Gamma \triangleq \left[\,\underline{\gamma}, \overline{\gamma}\,\right].$$

Under a boundedness assumption on t, may interchange max-min:

$$\begin{aligned}\widehat{\varphi}_\lambda(x) &= \min_{t\in\mathbb{R}} \max_{\gamma\in\Gamma} \mathbb{E}\left[\, H_\lambda(\psi(x;\widetilde{\omega});t;\gamma)\,\right] \\ &= \min_{t\in\mathbb{R}} \max\left\{\, \mathbb{E}\left[\, H_\lambda(\psi(x;\widetilde{\omega});t;\underline{\gamma})\,\right], \mathbb{E}\left[\, H_\lambda(\psi(x;\widetilde{\omega});t;\overline{\gamma})\,\right]\,\right\}.\end{aligned}$$

Special cases: (both with a nonsmooth H_λ)

- $H_\lambda(z;t;\gamma) \triangleq z + \lambda \left\{ t + \dfrac{1}{1-\gamma}\,[\,z-t\,]_+ \right\}$ and $\Gamma = \{\gamma\}$ lead to a mean-CVAR deviation function

- $H_\lambda(z;t;\gamma) \triangleq t + (\,1-\lambda+\lambda\,\gamma\,)\,(\,z-t\,) + \lambda\,[\,z-t\,]_+$ and $\Gamma = [0,1]$ lead to a mean-absolute semideviation function.

Assumptions

For each player $i = 1, \cdots, N$,

(H1): $H_{i;\lambda_i}(z; t_i; \gamma_i)$ is Lipschitz continuous; strictly increasing in z for fixed (t_i, γ_i); convex in (z, t_i) jointly for fixed γ_i; and linear in γ_i for fixed (z, t_i).

(H2): a constant $\eta_i > 0$ exists such that $|H_{i;\lambda_i}(z; t_i; \gamma_i)| \leq \eta_i(1 + \|z\|)$ for all $(t; \gamma_i) \in \mathbb{R} \times \Gamma_i$ and all z of interest.

(T): there exists a compact interval $\mathcal{T}_i$ such that the function $h_{i;\lambda_i}(x; t_i; \gamma_i) \triangleq \mathbb{E}\left[H_{i;\lambda_i}(\psi_i(x^i, x^{-i}; \widetilde{\omega}); t_i; \gamma_i) \right]$ satisfies

$$\underset{t_i \in \mathbb{R}}{\text{minimum}}\, h_{i;\lambda_i}(x; t_i; \gamma_i) = \underset{t_i \in \mathcal{T}_i}{\text{minimum}}\, h_{i;\lambda_i}(x; t_i; \gamma_i)$$

for all $x \in \mathbf{X}$ and every $\gamma_i \in \Gamma_i$.

Resulting in the game with non-differentiable, convex objectives:

$$\left\{ \begin{array}{l} \underset{x^i \in X^i,\, t_i \in \mathcal{T}_i}{\text{minimize}} \; \left[\theta_i(x^i, x^{-i}) + \right. \\ \quad \left. \max\left(\mathbb{E}\left[H_{i;\lambda_i}(\psi_i(x^i, x^{-i}; \widetilde{\omega}); t_i; \underline{\gamma}_i) \right], \mathbb{E}\left[H_{i;\lambda_i}(\psi_i(x^i, x^{-i}; \widetilde{\omega}); t_i; \overline{\gamma}_i) \right] \right) \right] \end{array} \right\}_{i=1}^{N}$$

Private recourse

$$\psi_i(x^i;\omega) \triangleq \underset{z^i}{\text{minimum}} \quad \left(f^{\,i}(\omega)\right)^T z^i + \tfrac{1}{2}\,(\,z^i\,)^T Q^{\,i} z^i$$
$$\text{subject to} \quad C^{\,ii}(\omega)x^i + D^{\,i}z^i \geq \xi^{\,i}(\omega)$$

leads to the following game:

$$\left\{ \underset{x^i \in X^i,\, t_i \in \mathcal{T}_i}{\text{minimize}} \quad \left[\theta_i(x^i, x^{-i}) + \underbrace{\varphi_i(x^i;t_i)}_{\text{private}} \right] \right\}_{i=1}^N,$$

where

$$\varphi_i(x^i;t_i) \triangleq \max\left(\mathbb{E}\left[H_{i;\lambda_i}(\psi_i(x^i;\widetilde{\omega});t_i;\underline{\gamma}_i) \right], \mathbb{E}\left[H_{i;\lambda_i}(\psi_i(x^i;\widetilde{\omega});t_i;\overline{\gamma}_i) \right] \right)$$

remains nonsmooth, albeit separable in players' individual variables.

Stochastic best-response: set-up

Given vector $y \in \mathbf{X}$ and tuples: L of positive sample sizes $\{L_i\}_{i=1}^N$, samples $\{\omega_j^i\}_{j=1}^{L_i}$, and positive scalars $\{p_j^i\}_{j=1}^{L_i}$ for each $i = 1, \cdots, L_i$, and with $\mathcal{T} \triangleq \prod_{i=1}^N \mathcal{T}_i$, let $\mathrm{BR}(y; L; \omega; p)$ be

$$\underset{x \in \mathbf{X};\, t \in \mathcal{T}}{\operatorname{argmin}} \left\{ \sum_{i=1}^N \left[\theta_i(x^i, y^{-i}) + \varphi_{i,L_i}(x^i; t_i; \omega^i; p^i) + \tfrac{1}{2} \| x^i - y^i \|^2 \right] \right\}$$

$$= \left\{ \underbrace{\underset{x^i \in X^i;\, t_i \in \mathcal{T}_i}{\operatorname{argmin}} \left[\theta_i(x^i, y^{-i}) + \varphi_{i,L_i}(x^i; t_i; \omega^i; p^i) + \tfrac{1}{2} \| x^i - y^i \|^2 \right]}_{\text{separable sample-based optimization in } (x^i, t_i)} \right\}_{i=1}^N ,$$

where $\varphi_{i,L_i}(x^i; t_i; \omega^i; p^i) \approx \varphi_i(x^i; t_i)$, such as a sample average approximation:

$$\max \left(\sum_{j=1}^{L_i} p_j^i \left[H_{i;\lambda_i}(\psi_i(x^i; \omega_j^i); t_i; \underline{\gamma}_i) \right], \sum_{j=1}^{L_i} p_j^i \left[H_{i;\lambda_i}(\psi_i(x^i; \omega_j^i); t_i; \overline{\gamma}_i) \right] \right).$$

Stochastic best-response

Joint sampling and best-response

Step 0. Set $\nu = 0$ and generate $L_{\nu;i}$ samples $\left\{\omega_1^{\nu,i}, \cdots, \omega_{L_{\nu;i}}^{\nu;i}\right\}$ with corresponding probabilities $\{p_1^{\nu;i}, \cdots, p_{L_i}^{\nu;i}\}$ for $i = 1, \cdots, N$.

Step 1. Solve BR$(x^\nu; L_\nu; \omega^\nu; p^\nu)$ for $i = 1, \cdots, N$.

Step 2. Set $\nu \leftarrow \nu + 1$. Update $\{L_{\nu;i}\}_{i=1}^N$, samples $\left\{\omega_j^{\nu;i}\right\}_{j=1}^{L_{\nu;i}}$, and scalars $\left\{p_j^{\nu;i}\right\}_{j=1}^{L_{\nu;i}}$. $\square$

• Proof of almost sure convergence is in progress; based on proof in deterministic case (contraction) and statistical properties of sampling approximation of expectation.

• Each best-response subproblem BR$(x^\nu; L_\nu; \omega^\nu; p^\nu)$ is a min-max program involving the (non-differentiable) sampled recourse function $\psi_i(x^i; \omega_j^i)$ and possible non-smoothness in the mean-deviation function $H_{i;\lambda_i}(\psi_i(x^i; \omega_j^i); t_i; \gamma_i)$.

Coupled second-stage constraints

Recall recourse given by dual program

$$\underset{z^i,\,v^i}{\text{maximize}} \quad \left[\xi^{\,i}(\omega) - \sum_{j=1}^{N} C^{\,ij}(\omega)x^j\right]^T v^i - \tfrac{1}{2}(\,z^i\,)^T Q^{\,i} z^i - \frac{s}{2}(\,v^i\,)^T v^i$$

$$\text{subject to} \quad -Q^{\,i}z^i + (\,D^{\,i}\,)^T v^i \,\le\, f^{\,i}(\omega) + \sum_{j\neq i} G^{\,ij}(\omega)x^j.$$

To ensure differentiability, regularize both primal and dual variables:

$$\underset{z^i,\,v^i}{\text{maximize}} \quad \left[\xi^{\,i}(\omega) - \sum_{j=1}^{N} C^{\,ij}(\omega)x^j\right]^T v^i - \tfrac{1}{2}(\,z^i\,)^T Q^{\,i} z^i - \frac{s}{2}\left[\,(\,z^i\,)^T z^i + (\,v^i\,)^T v^i\,\right]$$

$$\text{subject to} \quad -Q^{\,i}z^i + (\,D^{\,i}\,)^T v^i \,=\, f^{\,i}(\omega) + \sum_{j\neq i} G^{\,ij}(\omega)x^j$$

$$\text{and} \qquad v^i \,\ge\, 0.$$

Let $\psi_{s;i}(x;\omega)$ denote optimal objective value. For fixed $s > 0$, $\psi_{s;i}$ is Lipschitz and differentiable in x^i for all fixed (x^{-i},ω) such that constraint is feasible.

Smoothing the mean-deviation function

For $s > 0$, let

- $\max(a_1, a_2) \approx s \log\left[\exp(a_1/s) + \exp(a_2/s)\right]$ be an exponential smoothing of the pointwise max operator; and let

- $H_{s;i;\lambda_i} \approx H_{i;\lambda_i}$ be a smooth approximation of $H_{i;\lambda_i}$.

Leading to the following smoothed game:

$$\boxed{\left\{ \underset{x^i \in X^i,\, t_i \in \mathcal{T}_i}{\text{minimize}} \left(\theta_i(x^i, x^{-i}) + \widehat{h}^{\,\text{apprx}}_{i;\lambda_i;s}(x; t_i) + \frac{s}{2}\, t_i^2 \right) \right\}_{i=1}^{N},}$$

where $\widehat{h}^{\,\text{apprx}}_{i;\lambda_i;s}(x; t_i)$ is given by

$$s \log \left\{ \exp\left(\mathbb{E}\left[\frac{1}{s}\, H_{i;\lambda_i;s}(\psi_{i;s}(x; \widetilde{\omega}); t_i; \underline{\gamma}_i) \right] \right) + \exp\left(\mathbb{E}\left[\frac{1}{s}\, H_{i;\lambda_i;s}(\psi_{i;s}(x; \widetilde{\omega}); t_i; \overline{\gamma}_i) \right] \right) \right\}.$$

At this point, there remain two steps: sampling and best response.