Duality and optimality in stochastic optimization and mathematical finance

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- Stochastic optimization Duality
- Closedness criteria
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Conjugate duality

Stochastic optimization Duality Closedness criteria Optimality conditions Conjugate duality studies parametric optimization problems

minimize
$$F(x, u)$$
 over $x \in X$, (P)

where the parameter u takes values in a locally convex space U in separating duality with Y. If F is convex on $X \times U$, then

- the optimum value $\varphi(u)$ is convex on U,
- the associated Lagrangian

$$L(x,y) = \inf_{u \in U} \{ F(x,u) - \langle u, y \rangle \}.$$

is convex concave on $X \times Y$,

• the conjugate of φ can be expressed as

$$\varphi^*(y) = \sup_{u \in U} \{ \langle u, y \rangle - \varphi(u) \} = -\inf_{x \in X} L(x, y).$$

Conjugate duality

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• If φ is closed, then there is no duality gap: $\varphi = \varphi^{**}$.

 If φ is subdifferentiable at u, then x ∈ X solves (P) if and only if (x, y) is a saddle point of L − ⟨·, y⟩ for some y ∈ Y.

Note that we have not assumed that X is locally convex.

- In stochastic optimization, this will allow us to choose a large enough X so that primal solutions exist and φ is closed.
- On the other hand, we will have to work a bit harder to find explicit expressions for φ^* and the saddle point conditions.

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Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P)$ be a filtered probability space and consider the parametric optimization problem

minimize $Ef(x, u) := \int f(x(\omega), u(\omega), \omega) dP(\omega)$ over $x \in \mathcal{N}$,

- $\mathcal{N} = \{ (x_t)_{t=0}^T \mid x_t \in L^0(\Omega, \mathcal{F}_t, P; \mathbb{R}^d) \},\$
- $u \in L^p(\Omega, \mathcal{F}, P; \mathbb{R}^m)$ is a fixed parameter,
- $f: \mathbb{R}^{d(T+1)} \times \mathbb{R}^m \times \Omega \to \overline{\mathbb{R}}$ is a convex normal integrand,
- the integral is defined as $+\infty$ unless the positive part of the integrand is integrable.

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Example 1 (Inequality constraints) If

 $f(x, u, \omega) = \begin{cases} f_0(x, \omega) & \text{if } f_j(x, \omega) + u_j \leq 0 \text{ for } j = 1, \dots, m, \\ +\infty & \text{otherwise}, \end{cases}$

where f_j are convex normal integrands, then $\varphi(u)$ is the optimal value of the problem

 $\begin{array}{ll} \underset{x \in \mathcal{N}}{\text{minimize}} & Ef_0(x(\omega), \omega) \\ \text{subject to} & f_j(x(\omega), \omega) + u_j(\omega) \leq 0 \quad j = 1, \dots, m. \end{array}$

This was studied by [Rockafellar and Wets, 1978] in the case of bounded strategies.

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Example 2 (Shadow price of information) If m = d(T+1) and

 $f(x, u, \omega) = h(x + u, \omega),$

where h is a convex normal integrand, then the problem becomes the nonadapted perturbation

 $\underset{x \in \mathcal{N}}{\text{minimize}} \quad Eh(x+u).$

of the stochastic optimization problem

 $\underset{x \in \mathcal{N}}{\text{minimize}} \quad Eh(x).$

This was studied in [Rockafellar and Wets, 1976].

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Example 3 (Optimal stopping) If d = m = 1 and

$$f(x, u, \omega) = \begin{cases} \sum_{t=0}^{T} x_t Z_t(\omega) & \text{if } x \ge 0 \text{ and } \sum_{t=0}^{T} x_t \le u, \\ +\infty & \text{otherwise,} \end{cases}$$

for an adapted real-valued process Z, the problem becomes

$$\underset{x \in \mathcal{N}_{+}}{\text{minimize}} \quad E \sum_{t=0}^{T} x_{t} Z_{t} \quad \text{subject to} \quad \sum_{t=0}^{T} x_{t} \le u \ P\text{-a.s.}$$

When u = 1, this is a convex relaxation of the optimal stopping problem. The relaxation does not affect the optimal value.

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Example 4 (Optimal investment) Let m = 1 and

$$f(x, u, \omega) = v \left(u - \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}(\omega) \right)$$

where s is an adapted price process and $v : \mathbb{R} \to \overline{\mathbb{R}}$ is convex. The problem becomes

$$\underset{x \in \mathcal{N}}{\text{minimize}} \quad Ev\left(u - \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}\right)$$

which is the problem of optimal investment with liability u.

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- **Optimality conditions**

Example 5 (Optimal investment in illiquid markets) Let

$$f(x, u, \omega) = \begin{cases} \sum_{t=0}^{T} v_t(S_t(\Delta x_t, \omega) + u_t) & \text{if } x_t \in D_t(\omega), \ x_T = 0 \\ +\infty & \text{otherwise} \end{cases}$$

where

- $S_t : \mathbb{R}^d \times \Omega \to \mathbb{R}$ is such that $S_t(\cdot, \omega)$ are convex with $S_t(0, \omega) = 0$ and $S_t(x, \cdot)$ are \mathcal{F}_t -measurable,
- $\omega \mapsto D_t(\omega)$ is \mathcal{F}_t -measurable with $D_t(\omega)$ closed convex and $0 \in D_t(\omega)$,
- $v_t : \mathbb{R} \to \overline{\mathbb{R}}$ is convex.

The problem becomes

$$\underset{x \in \mathcal{N}_D}{\text{minimize}} \quad E \sum_{t=0}^T v_t (S_t(\Delta x_t) + u_t).$$

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We will apply conjugate duality with

$$F(x, u) = Ef(x, u),$$

$$X = \mathcal{N},$$

$$U = L^{p}(\Omega, \mathcal{F}, P; \mathbb{R}^{m}),$$

$$Y = L^{q}(\Omega, \mathcal{F}, P; \mathbb{R}^{m}),$$

$$\langle u, y \rangle = E(u \cdot y).$$

• Our first aim is to show that, one often has

$$\varphi^*(y) = -\inf_{x \in \mathcal{N}} L(x, y) = -\inf_{x \in \mathcal{N}^\infty} L(x, y)$$

where $\mathcal{N}^{\infty} := \mathcal{N} \cap L^{\infty}$.

• This yields explicit expressions for φ^* in many situations.

Let

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$$\begin{split} \tilde{\varphi}(u) &= \inf_{x \in \mathcal{N}^{\infty}} Ef(x, u), \\ \mathcal{N}^{\perp} &= \{ v \in L^{1}(\Omega, \mathcal{F}, P; \mathbb{R}^{n}) \,|\, E(x \cdot v) = 0 \,\,\forall x \in \mathcal{N}^{\infty} \}, \\ l(x, y, \omega) &= \inf_{u \in \mathbb{R}^{m}} \{ f(x, u, \omega) - u \cdot y \}. \end{split}$$

Theorem 1 If dom $El(\cdot, y) \cap \mathcal{N}^{\infty} \subseteq \operatorname{dom} \tilde{\varphi}$, then

$$\tilde{\varphi}^*(y) = -\inf_{x\in\mathcal{N}^\infty} El(x,y).$$

If in addition, there exists $v \in \mathcal{N}^{\perp}$ with $\tilde{\varphi}^*(y) = Ef^*(v, y)$, then

 $\varphi^*(y) = \tilde{\varphi}^*(y).$

Lemma 2 (Perkkiö, 2014) If $x \in \mathcal{N}$, $v \in \mathcal{N}^{\perp}$ and $E[x \cdot v]^+ \in L^1$, then $E(x \cdot v) = 0$.

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Example 6 (Inequality constraints) In the model

 $\underset{x \in \mathcal{N}}{\text{minimize}} \quad Ef_0(x(\omega), \omega)$

subject to $f_j(x(\omega), \omega) + u_j(\omega) \le 0$ j = 1, ..., m,

the conditions of Theorem 1 hold provided $f_j(x,\omega) \in L^p$ for all $x \in \mathbb{R}^n$. This follows from the Mackey-continuity of $El(\cdot, y)$ on L^∞ , which in turn follows from Rockafellar, Integrals which are convex functionals II, 1971. We have

$$l(x, y, \omega) = \begin{cases} f_0(x, \omega) + \sum_{j=1}^m y_j f_j(x, \omega) & \text{if } y \ge 0, \\ -\infty & \text{otherwise} \end{cases}$$

but, in general, don't have explicit expressions for $\varphi^*(y)$.

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$$\varphi^*(y) = \begin{cases} Eh^*(y) & \text{if } E_t y_t = 0 \ \forall t, \\ \infty & \text{otherwise} \end{cases}$$

and, in particular,

$$\varphi^{**}(0) = \sup_{y \in \mathcal{N}^{\perp}} E[-h^*(y)] = \sup_{y \in \mathcal{N}^{\perp}} E\inf_{x \in \mathbb{R}^n} \{h(x) - x \cdot y\},\$$

hence the name shadow price of information; see [Rockafellar and Wets, 1976], [Back and Pliska, 1987] and [Davis, 1992]. This allows for MC much as in [Rogers, 2002].

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$$\begin{array}{ll} \underset{x \in \mathcal{N}_{+}}{\operatorname{maximize}} & E \sum_{t=0}^{T} x_{t} Z_{t} \quad \text{subject to} \quad \sum_{t=0}^{T} x_{t} \leq 1 \\ \\ equals \ that \ of \\ \underset{y \in \mathcal{M}^{\infty}}{\operatorname{minimize}} & y_{0} \quad \text{subject to} \quad y \geq Z^{+}, \\ \\ where \ \mathcal{M}^{\infty} \ is \ the \ set \ of \ bounded \ martingales \ and \ Z^{+} \ is \ the \\ positive \ part \ of \ Z. \end{array}$$

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$$\varphi^{**}(u) = \sup_{y \in \mathcal{Q}} E[uy - V^*(y)],$$

where Q is the set of positive multiples of martingale densities $y \in \mathcal{Y}$, i.e. densities dQ/dP of probability measures $Q \ll P$ under which the price process s is a martingale.

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 The above expressions for φ^{**} provide dual representations of the optimal value φ provided φ is proper and lower semicontinuous (lsc), i.e.

 $\liminf_{\nu \to \infty} \varphi(u^{\nu}) \ge \varphi(u)$

whenever $u^{\nu} \rightarrow u$ in L^p .

- The traditional "direct method" assumes that Ef is jointly lsc and $Ef(\cdot, u)$ is inf-compact uniformly in u.
- In financial models, the topological inf-compactness condition often fails but there is a more general measure theoretic counterpart that works well in \mathcal{N} .

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Theorem 3 (Komlós) If $(x^{\nu})_{\nu=1}^{\infty} \subset L^0(\Omega, \mathcal{F}, P; \mathbb{R}^n)$ is almost surely bounded in the sense that

 $\sup_{\nu} |x^{\nu}(\omega)| < \infty \quad P\text{-a.s.}$

then there is a sequence of convex combinations $\bar{x}^{\nu} \in \operatorname{co}\{x^{\mu} \mid \mu \geq \nu\}$ that converges almost surely in L^{0} .

This yields the following infinite-dimensional version of Theorem 8.4 from Convex Analysis.

Theorem 4 Let $C : \Omega \rightrightarrows \mathbb{R}^n$ be closed convex-valued and \mathcal{F} -measurable. If $\{x \in \mathcal{N} \mid x \in C^\infty \ a.s.\} = \{0\}$, then every sequence in $\{x \in \mathcal{N} \mid x \in C \ a.s.\}$ is almost surely bounded.

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Theorem 5 Assume that f is bounded from below and that $\{x \in \mathcal{N} | f^{\infty}(x(\omega), 0, \omega) \leq 0 \text{ a.s.}\}$

is a linear space. Then

 $\varphi(u) = \inf_{x \in \mathcal{N}} Ef(x, u)$

is closed on L^p and the inf is attained for every $u \in L^p$.

The lower bound has been relaxed in [Perkkiö, 2014].

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Example 10 (Optimal stopping) When

$$f(x, u, \omega) = \begin{cases} -\sum_{t=0}^{T} x_t Z_t(\omega) & \text{if } x \ge 0 \text{ and } \sum_{t=0}^{T} x_t \le u, \\ +\infty & \text{otherwise,} \end{cases}$$

we have $f^{\infty} = f$ and

$$\{x \in \mathcal{N} | f^{\infty}(x, 0) \le 0 \text{ a.s.}\} = \{0\},\$$

so the linearity condition is always satisfied.

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Example 11 (Shadow price of information) When

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f(x, u, \omega) = h(x + u, \omega),
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the linearity condition means that

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\{x \in \mathcal{N} \mid h^{\infty}(x) \le 0 \text{ -a.s.}\}
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is linear.

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Example 12 (Optimal investment) When $f(x, u, \omega) = \begin{cases} v \left(u - \sum_{t=0}^{T-1} x_t \cdot \Delta S_{t+1}(\omega) \right) & \text{if } x_t \in D_t(\omega), \\ +\infty & \text{otherwise} \end{cases}$

we get

$$f^{\infty}(x, u, \omega) = \begin{cases} v^{\infty} \left(u - \sum_{t=0}^{T-1} x_t \cdot \Delta S_{t+1}(\omega) \right) & \text{if } x_t \in D_t^{\infty}(\omega) \\ +\infty & \text{otherwise.} \end{cases}$$

If v is nonconstant and $D_t(\omega) = \mathbb{R}^J$, the linearity condition becomes the no-arbitrage condition

$$x \in \mathcal{N}: \sum x_t \cdot \Delta S_{t+1} \ge 0 \implies \sum x_t \cdot \Delta S_{t+1} = 0.$$

Example 13 With transaction costs, we get the robust no-arbitrage condition introduced by [Schachermayer, 2004].

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The linearity condition may hold even under arbitrage. **Example 14** It holds if $S_t^{\infty}(x, \omega) > 0$ for $x \notin \mathbb{R}^J_-$. **Example 15** In [Çetin and Rogers, 2007] with $S_t(x, \omega) = x^0 + s_t(\omega)\psi(x^1)$

one has $S_t^{\infty}(x, \omega) = x^0 + s_t(\omega)\psi^{\infty}(x^1)$. When $\inf \psi' = 0$ and $\sup \psi' = \infty$ we have $\psi^{\infty} = \delta_{\mathbb{R}_-}$, so the condition in Example 14 holds.

Example 16 If $S_t(\cdot, \omega) = s_t(\omega) \cdot x$ for a componentwise strictly positive price process s and $D_t^{\infty}(\omega) \subseteq \mathbb{R}^J_+$ (infinite short selling is prohibited) then linearity condition holds.

Optimality conditions

Stochastic optimization Duality Closedness criteria Optimality conditions **Theorem 6** Assume that $\partial \varphi(u) \neq \emptyset$ and that for every $y \in \partial \varphi(u)$ there exists $v \in \mathcal{N}^{\perp}$ such that $\varphi^*(y) = Ef^*(v, y)$. Then an $x \in \mathcal{N}$ solves (P) if and only if it is feasible and there exist $y \in \mathcal{Y}$ and $v \in \mathcal{N}^{\perp}$ such that

 $(v,y) \in \partial f(x,u)$

P-almost surely, or equivalently, if

 $v \in \partial_x l(x, y)$ and $u \in \partial_y [-l](x, y)$

P-almost surely.

Optimality conditions

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 $f_j(x) + u_j \leq 0,$ $x \in \underset{z \in \mathbb{R}^n}{\operatorname{argmin}} \{ f_0(z) + \sum_{j=1}^m y_j f_j(z) - z \cdot v \},$ $y_j f_j(z) = 0 \quad j = 1, \dots, m,$ $y_j \geq 0$

P-almost surely. [Rockafellar and Wets, 1978] give sufficient conditions for the existence of an optimal $x \in \mathcal{N}^{\infty}$ and the corresponding dual variables $y \in \mathcal{Y}$ and $v \in \mathcal{N}^{\perp}$.

Optimality conditions

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Example 18 Consider the optimal investment problem in liquid markets and assume that EV is finite on L^p and EV^* is proper in L^q . Then an $x \in \mathcal{N}$ solves (P) if and only if it is feasible and there exists a martingale density y for s such that

$$y \in \partial V(u - \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1})$$
 P-a.s.

much like e.g. in [Schachermayer, 2001] or [Biagini and Frittelli, 2008] in continuous time.

Continuous-time models

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- Continuous time models of financial mathematics are often expressed in terms of the stochastic integral (wealth process) of the portfolio process wrt the price process.
- However, the definition of $(x, u) \mapsto Ef(x, u)$ relies on scenariowise description of f(x, u).
- Our approach is first to consider strategies of bounded variation and then take the lsc hull of the the objective Ef
 - For BV strategies, stochastic integrals are given pathwise and the dual can often be written down explicitly.
 - \circ Taking the lsc hull of Ef gives rise to stochastic integrals and it does not affect the dual problem.
 - Does require conjugation of integral functionals on spaces of stochastic processes.