Duality and optimality in stochastic optimization and mathematical finance

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Conjugate duality

Conjugate duality studies parametric optimization problems

$$\text{minimize} \quad F(x,u) \quad \text{over} \quad x \in X, \tag{P}$$

where the parameter u takes values in a locally convex space U in separating duality with Y. If F is convex on $X \times U$, then

- the optimum value $\varphi(u)$ is convex on U,
- the associated Lagrangian

  $$L(x,y) = \inf_{u \in U}\{F(x,u) - \langle u, y\rangle\}.$$

  is convex concave on $X \times Y$,
- the conjugate of φ can be expressed as

  $$\varphi^*(y) = \sup_{u \in U}\{\langle u, y\rangle - \varphi(u)\} = -\inf_{x \in X} L(x,y).$$

Conjugate duality

- If φ is closed, then there is no duality gap: $\varphi = \varphi^{**}$.
- If φ is subdifferentiable at u, then $x \in X$ solves (P) if and only if (x, y) is a saddle point of $L - \langle \cdot, y \rangle$ for some $y \in Y$.

Note that we have not assumed that X is locally convex.

- In stochastic optimization, this will allow us to choose a large enough X so that primal solutions exist and φ is closed.
- On the other hand, we will have to work a bit harder to find explicit expressions for φ^* and the saddle point conditions.

Stochastic optimization

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P)$ be a filtered probability space and consider the parametric optimization problem

$$\text{minimize } Ef(x,u) := \int f(x(\omega), u(\omega), \omega) dP(\omega) \text{ over } x \in \mathcal{N},$$

- $\mathcal{N} = \{(x_t)_{t=0}^T \,|\, x_t \in L^0(\Omega, \mathcal{F}_t, P; \mathbb{R}^d)\}$,
- $u \in L^p(\Omega, \mathcal{F}, P; \mathbb{R}^m)$ is a fixed parameter,
- $f : \mathbb{R}^{d(T+1)} \times \mathbb{R}^m \times \Omega \to \overline{\mathbb{R}}$ is a convex normal integrand,
- the integral is defined as $+\infty$ unless the positive part of the integrand is integrable.

Example 1 (Inequality constraints) *If*

$$f(x,u,\omega)=\begin{cases} f_0(x,\omega) & \textit{if } f_j(x,\omega)+u_j\le 0 \textit{ for } j=1,\dots,m,\\ +\infty & \textit{otherwise,}\end{cases}$$

where f_j are convex normal integrands, then $\varphi(u)$ is the optimal value of the problem

$$\begin{aligned} &\underset{x\in\mathcal{N}}{\text{minimize}} && Ef_0(x(\omega),\omega)\\ &\text{subject to} && f_j(x(\omega),\omega)+u_j(\omega)\le 0 \quad j=1,\dots,m. \end{aligned}$$

This was studied by [Rockafellar and Wets, 1978] in the case of bounded strategies.

Example 2 (Shadow price of information) *If* $m = d(T+1)$ *and*

$$f(x, u, \omega) = h(x + u, \omega),$$

where h *is a convex normal integrand, then the problem becomes the nonadapted perturbation*

$$\underset{x \in \mathcal{N}}{\text{minimize}} \quad Eh(x + u).$$

of the stochastic optimization problem

$$\underset{x \in \mathcal{N}}{\text{minimize}} \quad Eh(x).$$

This was studied in [Rockafellar and Wets, 1976].

Example 3 (Optimal stopping) *If $d = m = 1$ and*

$$f(x,u,\omega) = \begin{cases} \sum_{t=0}^{T} x_t Z_t(\omega) & \text{if } x \geq 0 \text{ and } \sum_{t=0}^{T} x_t \leq u, \\ +\infty & \text{otherwise,} \end{cases}$$

for an adapted real-valued process Z, the problem becomes

$$\underset{x \in \mathcal{N}_+}{\text{minimize}} \quad E \sum_{t=0}^{T} x_t Z_t \quad \text{subject to} \quad \sum_{t=0}^{T} x_t \leq u \; P\text{-a.s.}$$

When $u = 1$, this is a convex relaxation of the optimal stopping problem. The relaxation does not affect the optimal value.

Example 4 (Optimal investment) *Let* $m = 1$ *and*

$$f(x,u,\omega) = v\left(u - \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}(\omega)\right)$$

where s *is an adapted price process and* $v : \mathbb{R} \to \overline{\mathbb{R}}$ *is convex. The problem becomes*

$$\underset{x \in \mathcal{N}}{\text{minimize}} \quad Ev\left(u - \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}\right)$$

which is the problem of optimal investment with liability u*.*

Stochastic optimization

Example 5 (Optimal investment in illiquid markets) *Let*

$$f(x,u,\omega)=\begin{cases}\sum_{t=0}^T v_t(S_t(\Delta x_t,\omega)+u_t) & \text{if } x_t\in D_t(\omega),\ x_T=0\\ +\infty & \text{otherwise}\end{cases}$$

where

- $S_t:\mathbb{R}^d\times\Omega\to\mathbb{R}$ *is such that* $S_t(\cdot,\omega)$ *are convex with* $S_t(0,\omega)=0$ *and* $S_t(x,\cdot)$ *are* $\mathcal{F}_t$*-measurable,*
- $\omega\mapsto D_t(\omega)$ *is* $\mathcal{F}_t$*-measurable with* $D_t(\omega)$ *closed convex and* $0\in D_t(\omega)$*,*
- $v_t:\mathbb{R}\to\overline{\mathbb{R}}$ *is convex.*

The problem becomes

$$\underset{x\in\mathcal{N}_D}{\text{minimize}}\quad E\sum_{t=0}^T v_t(S_t(\Delta x_t)+u_t).$$

Duality

We will apply conjugate duality with

$$\begin{aligned} F(x,u) &= Ef(x,u), \\ X &= \mathcal{N}, \\ U &= L^p(\Omega,\mathcal{F},P;\mathbb{R}^m), \\ Y &= L^q(\Omega,\mathcal{F},P;\mathbb{R}^m), \\ \langle u,y\rangle &= E(u\cdot y). \end{aligned}$$

- Our first aim is to show that, one often has

$$\varphi^*(y) = -\inf_{x\in\mathcal{N}} L(x,y) = -\inf_{x\in\mathcal{N}^\infty} L(x,y)$$

  where $\mathcal{N}^\infty := \mathcal{N}\cap L^\infty$.
- This yields explicit expressions for φ^* in many situations.

Duality

Let

$$\tilde{\varphi}(u) = \inf_{x\in\mathcal{N}^\infty} Ef(x,u),$$
$$\mathcal{N}^\perp = \{v \in L^1(\Omega,\mathcal{F},P;\mathbb{R}^n) \mid E(x\cdot v) = 0\ \forall x \in \mathcal{N}^\infty\},$$
$$l(x,y,\omega) = \inf_{u\in\mathbb{R}^m}\{f(x,u,\omega) - u\cdot y\}.$$

Theorem 1 *If* $\operatorname{dom} El(\cdot,y)\cap\mathcal{N}^\infty \subseteq \operatorname{dom}\tilde{\varphi}$, *then*

$$\tilde{\varphi}^*(y) = -\inf_{x\in\mathcal{N}^\infty} El(x,y).$$

If in addition, there exists $v\in\mathcal{N}^\perp$ *with* $\tilde{\varphi}^*(y) = Ef^*(v,y)$, *then*

$$\varphi^*(y) = \tilde{\varphi}^*(y).$$

Lemma 2 (Perkkiö, 2014) *If* $x\in\mathcal{N}$, $v\in\mathcal{N}^\perp$ *and* $E[x\cdot v]^+\in L^1$, *then* $E(x\cdot v) = 0$.

Duality

Example 6 (Inequality constraints) *In the model*

$$\begin{aligned} \operatorname*{minimize}_{x\in\mathcal{N}} \quad & Ef_0(x(\omega),\omega) \\ \text{subject to} \quad & f_j(x(\omega),\omega)+u_j(\omega)\le 0 \quad j=1,\dots,m, \end{aligned}$$

the conditions of Theorem 1 hold provided $f_j(x,\omega)\in L^p$ for all $x\in\mathbb{R}^n$. This follows from the Mackey-continuity of $El(\cdot,y)$ on L^∞, which in turn follows from Rockafellar, Integrals which are convex functionals II, *1971. We have*

$$l(x,y,\omega)=\begin{cases} f_0(x,\omega)+\sum_{j=1}^m y_j f_j(x,\omega) & \textit{if } y\ge 0,\\ -\infty & \textit{otherwise}\end{cases}$$

but, in general, don't have explicit expressions for $\varphi^*(y)$.

Duality

Example 7 (Shadow price of information) *When $f(x,u,\omega) = h(x+u,\omega)$, the conditions of Theorem 1 hold as soon as $Eh \cap L^p$. We get*

$$\varphi^*(y) = \begin{cases} Eh^*(y) & \text{if } E_t y_t = 0 \ \forall t, \\ \infty & \text{otherwise} \end{cases}$$

and, in particular,

$$\varphi^{**}(0) = \sup_{y \in \mathcal{N}^\perp} E[-h^*(y)] = \sup_{y \in \mathcal{N}^\perp} E \inf_{x \in \mathbb{R}^n} \{h(x) - x \cdot y\},$$

hence the name shadow price of information; see [Rockafellar and Wets, 1976], [Back and Pliska, 1987] and [Davis, 1992]. This allows for MC much as in [Rogers, 2002].

Duality

Example 8 (Optimal stopping) *The optimal value of the optimal stopping problem*

$$\underset{x\in\mathcal{N}_+}{\text{maximize}} \quad E\sum_{t=0}^{T} x_t Z_t \quad \text{subject to} \quad \sum_{t=0}^{T} x_t \le 1$$

equals that of

$$\underset{y\in\mathcal{M}^\infty}{\text{minimize}} \quad y_0 \quad \text{subject to} \quad y \ge Z^+,$$

where $\mathcal{M}^\infty$ is the set of bounded martingales and Z^+ is the positive part of Z.

Duality

Example 9 (Optimal investment) *Assume that,* $\forall x \in \mathcal{N}^\infty$ $\exists u \in L^p$ *such that* $EV(u - \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}) < \infty$, *then*

$$\varphi^{**}(u) = \sup_{y \in \mathcal{Q}} E[uy - V^*(y)],$$

where $\mathcal{Q}$ *is the set of positive multiples of martingale densities* $y \in \mathcal{Y}$, *i.e. densities* dQ/dP *of probability measures* $Q \ll P$ *under which the price process* s *is a martingale.*

Closedness criteria

- The above expressions for φ^{**} provide dual representations of the optimal value φ provided φ is proper and lower semicontinuous (lsc), i.e.

$$\liminf_{\nu\to\infty} \varphi(u^\nu) \geq \varphi(u)$$

whenever $u^\nu \to u$ in L^p.

- The traditional "direct method" assumes that Ef is jointly lsc and $Ef(\cdot, u)$ is inf-compact uniformly in u.
- In financial models, the topological inf-compactness condition often fails but there is a more general measure theoretic counterpart that works well in $\mathcal{N}$.

Closedness criteria

Theorem 3 (Komlós) *If $(x^\nu)_{\nu=1}^\infty \subset L^0(\Omega, \mathcal{F}, P; \mathbb{R}^n)$ is almost surely bounded in the sense that*

$$\sup_\nu |x^\nu(\omega)| < \infty \quad P\text{-a.s.}$$

then there is a sequence of convex combinations $\bar{x}^\nu \in \mathrm{co}\{x^\mu \,|\, \mu \geq \nu\}$ that converges almost surely in L^0.

This yields the following infinite-dimensional version of Theorem 8.4 from Convex Analysis.

Theorem 4 *Let $C : \Omega \rightrightarrows \mathbb{R}^n$ be closed convex-valued and $\mathcal{F}$-measurable. If $\{x \in \mathcal{N} \,|\, x \in C^\infty \ a.s.\} = \{0\}$, then every sequence in $\{x \in \mathcal{N} \,|\, x \in C \ a.s.\}$ is almost surely bounded.*

Closedness criteria

Theorem 5 *Assume that f is bounded from below and that*

$$\{x \in \mathcal{N} \,|\, f^\infty(x(\omega), 0, \omega) \le 0 \text{ a.s.}\}$$

is a linear space. Then

$$\varphi(u) = \inf_{x \in \mathcal{N}} Ef(x, u)$$

is closed on L^p and the inf is attained for every $u \in L^p$.

The lower bound has been relaxed in [Perkkiö, 2014].

Example 10 (Optimal stopping) *When*

$$f(x,u,\omega)=\begin{cases}-\sum_{t=0}^T x_t Z_t(\omega) & \text{if } x\ge 0 \text{ and } \sum_{t=0}^T x_t\le u,\\ +\infty & \text{otherwise,}\end{cases}$$

we have $f^\infty = f$ *and*

$$\{x\in\mathcal{N} \mid f^\infty(x,0)\le 0 \text{ a.s.}\}=\{0\},$$

so the linearity condition is always satisfied.

Closedness criteria

Example 11 (Shadow price of information) *When*

$$f(x,u,\omega) = h(x+u,\omega),$$

the linearity condition means that

$$\{x \in \mathcal{N} \,|\, h^\infty(x) \le 0 \text{ -a.s.}\}$$

is linear.

Closedness criteria

Example 12 (Optimal investment) *When*

$$f(x,u,\omega) = \begin{cases} v\left(u - \sum_{t=0}^{T-1} x_t \cdot \Delta S_{t+1}(\omega)\right) & \text{if } x_t \in D_t(\omega), \\ +\infty & \text{otherwise} \end{cases}$$

we get

$$f^\infty(x,u,\omega) = \begin{cases} v^\infty\left(u - \sum_{t=0}^{T-1} x_t \cdot \Delta S_{t+1}(\omega)\right) & \text{if } x_t \in D_t^\infty(\omega) \\ +\infty & \text{otherwise.} \end{cases}$$

If v is nonconstant and $D_t(\omega) = \mathbb{R}^J$, the linearity condition becomes the no-arbitrage condition

$$x \in \mathcal{N} : \sum x_t \cdot \Delta S_{t+1} \geq 0 \implies \sum x_t \cdot \Delta S_{t+1} = 0.$$

Example 13 *With transaction costs, we get the robust no-arbitrage condition introduced by [Schachermayer, 2004].*

Closedness criteria

The linearity condition may hold even under arbitrage.

Example 14 *It holds if* $S_t^\infty(x,\omega) > 0$ *for* $x \notin \mathbb{R}^J_-$.

Example 15 *In [Çetin and Rogers, 2007] with*

$$S_t(x,\omega) = x^0 + s_t(\omega)\psi(x^1)$$

one has $S_t^\infty(x,\omega) = x^0 + s_t(\omega)\psi^\infty(x^1)$. *When* $\inf\psi' = 0$ *and* $\sup\psi' = \infty$ *we have* $\psi^\infty = \delta_{\mathbb{R}_-}$, *so the condition in Example 14 holds.*

Example 16 *If* $S_t(\cdot,\omega) = s_t(\omega)\cdot x$ *for a componentwise strictly positive price process* s *and* $D_t^\infty(\omega) \subseteq \mathbb{R}^J_+$ *(infinite short selling is prohibited) then linearity condition holds.*

Optimality conditions

Theorem 6 *Assume that* $\partial\varphi(u) \neq \emptyset$ *and that for every* $y \in \partial\varphi(u)$ *there exists* $v \in \mathcal{N}^\perp$ *such that* $\varphi^*(y) = Ef^*(v, y)$*. Then an* $x \in \mathcal{N}$ *solves* (P) *if and only if it is feasible and there exist* $y \in \mathcal{Y}$ *and* $v \in \mathcal{N}^\perp$ *such that*

$$(v, y) \in \partial f(x, u)$$

P*-almost surely, or equivalently, if*

$$v \in \partial_x l(x, y) \quad \textit{and} \quad u \in \partial_y[-l](x, y)$$

P*-almost surely.*

Optimality conditions

Example 17 *In the model with constraints, the optimality condition of Theorem 6 means that*

$$\begin{aligned} f_j(x) + u_j &\le 0, \\ x \in \operatorname*{argmin}_{z\in\mathbb{R}^n}\{f_0(z) + \sum_{j=1}^m y_j f_j(z) - z\cdot v\}&, \\ y_j f_j(z) = 0 &\quad j = 1,\dots,m, \\ y_j &\ge 0 \end{aligned}$$

P-almost surely. [Rockafellar and Wets, 1978] give sufficient conditions for the existence of an optimal $x \in \mathcal{N}^\infty$ and the corresponding dual variables $y \in \mathcal{Y}$ and $v \in \mathcal{N}^\perp$.

Optimality conditions

Example 18 *Consider the optimal investment problem in liquid markets and assume that EV is finite on L^p and EV^* is proper in L^q. Then an $x \in \mathcal{N}$ solves (P) if and only if it is feasible and there exists a martingale density y for s such that*

$$y \in \partial V(u - \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}) \quad P\text{-a.s.}$$

much like e.g. in [Schachermayer, 2001] or [Biagini and Frittelli, 2008] in continuous time.

Continuous-time models

- Continuous time models of financial mathematics are often expressed in terms of the stochastic integral (wealth process) of the portfolio process wrt the price process.
- However, the definition of $(x, u) \mapsto Ef(x, u)$ relies on scenariowise description of $f(x, u)$.
- Our approach is first to consider strategies of bounded variation and then take the lsc hull of the the objective Ef
  - For BV strategies, stochastic integrals are given pathwise and the dual can often be written down explicitly.
  - Taking the lsc hull of Ef gives rise to stochastic integrals and it does not affect the dual problem.
  - Does require conjugation of integral functionals on spaces of stochastic processes.