Regularity in variational analysis

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1 A general opinion

Nonsmooth analysis is a mess

A fact:

Nonsmooth analysis is part of Variational Analysis

Conclusion:

Variational Analysis is a mess!

2 Some of the main concepts of nonsmooth analysis

Some subdifferentials for $f : X \to \overline{\mathbb{R}}$ finite at x :

The simplest and the smallest one: the *firm* or *Fréchet subdifferential* $\partial_F f$:

$$x^* \in \partial_F f(x) \Longleftrightarrow \liminf_{u \to 0} (1/\|u\|)[f(x+u) - f(x) - \langle x^*, u\rangle] \geq 0$$
$$\Longleftrightarrow \exists b : X \to X^* : \ f(w) - f(x) \geq b(w).(w - x), \quad x^* = \lim_{w \to x} b(w).$$

The *directional* or *Dini-Hadamard subdifferential* $\partial_D f(x)$:

$$x^* \in \partial_D f(x) \Longleftrightarrow \langle x^*, u\rangle \leq r \quad \forall (u, r) \in T^D(\mathrm{epi} f, (x, f(x))$$
$$\Longleftrightarrow \forall u \in X \quad f^D(x, u) := \liminf_{(t,v) \to (0_+, u)} (1/t)[f(x + tv) - f(x)] \geq \langle x^*, u\rangle.$$

where $T^D(E, z)$ is the *contingent cone* to the epigraph E of f at $z := (x, f(x))$:

$$T^D(E, z) := \{w : \exists (t_n) \to 0_+, (w_n) \to w, \ z + t_n w_n \in E\}$$

The *incident* (or intermediate or adjacent) subdifferential $\partial_I f$

$$x^* \in \partial_I f(x) \Longleftrightarrow \forall u \in X \quad \langle x^*, u\rangle \leq f^I(x,u) := \inf\{r : (u,r) \in T^I(E,z)\}$$

where $T^I(E,z)$ the *incident tangent cone* to the epigraph E of f at $z := (x, f(x))$ or the set of velocities of moving points in E :

$$w \in T^I(E,z) \Longleftrightarrow \lim_{t\to 0_+} \frac{1}{t} d(z+tw, E) = 0 \Longleftrightarrow \exists c : \mathbb{R}_+ \to E,\ c(0) = z, \quad c'(0) = w.$$

The *circa-subdifferential* or *Clarke subdifferential* $\partial_C f(x)$:

$$x^* \in \partial_C f(x) \Longleftrightarrow \forall u \in X \quad \langle x^*, u\rangle \leq f^C(x,u) := \inf\{r : (u,r) \in T^C(E,z)\},$$

where $T^C(E,z)$ is the *Clarke* or *circa-tangent cone* to E at z.

The *limiting subdifferential* $\partial_L f(x)$:

$$x^* \in \partial_L f(x) \Longleftrightarrow \exists\ (x_n) \to x \text{ s.t. } (f(x_n)) \to f(x),\ x_n^* \in \partial_F f(x_n), (x_n^*) \rightharpoondown x^*.$$

3 Some simple properties

These subdifferentials satisfy common calculus rules.

They also have different properties and it may be sensible to try to combine them:

"elementary subdifferentials" ∂_D, ∂_F, ∂_I have nice order properties

∂_C and ∂_L are more efficient in terms of calculus rules.

Still, the latter enjoy some order properties
and the former dispose of some calculus rules such as

$$\partial(f+g)(x) \subset \partial f(x) \boxminus \partial(-g)(x)$$

where, for A, $B \subset X^*$ one sets

$$A \boxminus B := \{x^* \in X^* : x^* + B \subset A\}.$$

3.1 Semi-separable functions

Proposition 1 *Let* W, X *be normed spaces,* $g : W \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$, $h : X \to \mathbb{R}_\infty$, $A \in L(W, X)$, $f : W \times X \to \mathbb{R}_\infty$ *given by* $f(w, x) := g(w) + h(Bw + x)$. *Then*

$$(w^*, x^*) \in \partial f(w, x) \Leftrightarrow \exists u^* \in \partial g(w), \exists x^* \in \partial h(Aw + x), (w^*, x^*) = (u^* + B^* x^*, x^*).$$

3.2 Partial conjugacy

Given $f : X \times V \to \overline{\mathbb{R}}$ finite at (x, v) we denote by $f^{\ltimes}$ the conjugate of f with respect to its second variable: for $f_x := f(x, \cdot)$

$$f^{\ltimes}(x, y) := \sup_{v \in V} \langle y, v \rangle - f(x, v) := (f_x)^*(y)$$

Theorem 2 ***(Ioffe)*** *If* f *is l.s.c., if for all* $x \in X$ *the function* f_x *is convex and if* $x \rightrightarrows \operatorname{epi} f_x$ *is pseudo-Lipschitzian around* $(x, (x, f(x, v)))$, *then*

$$(x^*, y) \in \partial_L f(x, v) \Rightarrow -x^* \in \operatorname{co}\{w^* : (w^*, v) \in \partial_L f^{\ltimes}(x, y)\} \Rightarrow (-x^*, v) \in \partial_C f^{\ltimes}(x, y).$$

Proposition 3 *If for some* $x \in X$ *the function* f_x *is convex and if* $y \in \partial f_x(v)$, $x^* \in \widetilde{\partial}_F f_v(x) := -\partial_F(-f_v)(x)$ *then*

$$(-x^*, v) \in \partial_F f^{\ltimes}(x, y).$$

Proof. Let r be a remainder i.e. such that $r(u)/\|u\| \to 0$ as $\|u\| \to 0_+$ satisfying

$$f_v(x+u) - f_v(x) - \langle x^*, u\rangle \le r(u).$$

Then

$$\begin{aligned} f^{\ltimes}(x+u, y+w) - f^{\ltimes}(x,y) &\ge \langle y+w, v\rangle - f(x+u, v) + \langle y, v\rangle - f(x,v) \\ &\ge \langle w, v\rangle + \langle -x^*, u\rangle - r(u) \end{aligned}$$

and that shows that

$$(-x^*, v) \in \partial_F f^{\ltimes}(x,y).$$

The word “regularity” has many different meanings in mathematics.

Given two subdifferentials ∂_A and ∂_B, we say that a function f is *A-B-regular at some point* x of its domain if $\partial_A f(x) = \partial_B f(x)$.

Of course, if f is convex or approximately convex, or of class C^1, f is A-B-regular for all usual subdifferentials ∂_A and ∂_B.

D-I regularity=Rockafellar’s protodifferentiability” or “epidifferentiability”

C-D regularity= Clarke regularity

C-L regularity= and J.-P. P’s softness.

Since the calculus rules for ∂_A and ∂_B may be different, in some cases of interest A-B-regularity can be transferred to new functions build from f.

The purposes of such concepts are twofold: first, for restricted classes of functions, they reduce the number of available subdifferentials; second, they enable to get new properties, in particular equalities instead of inclusions.

In this talk we want to transfer regularity of an integrand to regularity of the integral functional.

We rely on the impressive work of E. Giner .

The proofs use the following simple geometric property.

Lemma 4 *(a) If $A : X \to Y$ is a continuous linear map between two normed spaces and x is a point of a subset E of X, then one has*

$$A(T^I(E,x)) \subset T^I(A(E), Ax),$$
$$A(T^D(E,x)) \subset T^D(A(E), Ax).$$

(b) If A is open from E onto $A(E)$ at x, in the sense that for any sequence (y_n) of $A(E)$ with limit Ax there exists a sequence $(x_n) \to x$ in E such that $Ax_n = y_n$ for all n, then

$$A(T^C(E,x)) \subset T^C(A(E), Ax).$$

A can be replaced with a differentiable (resp. circa-differentiable) map.

Corollary 5 *If U, V are normed spaces, if $h : U \to V$ is differentiable at $u \in U$, if $g : V \to \overline{\mathbb{R}}$ is finite at $v := h(u)$, and if $f := g \circ h$, then for all $u' \in U$ one has*

$$g^I(v, h'(u)u') \leq f^I(u, u'),$$
$$g^D(v, h'(u)u') \leq f^D(u, u').$$

4 Preliminaries about integral functionals

Let us recall that, given a measure space $(S, \mathcal{S}, \mu)$, the (upper) integral of a measurable function $u : S \to \overline{\mathbb{R}}$ is defined by

$$\begin{aligned} I(u) &:= \int u d\mu := \inf\{\int_S y(s)ds : \ y \in L_1(S), \ u(\cdot) \leq y(\cdot) \text{ a.e.}\} \\ &= \int \max(u,0)d\mu + \int \min(u,0)d\mu \end{aligned}$$

Given $f : S \times E \to \mathbb{R}_\infty$ the associated *integral functional* $F := I_f$ on $L_p(S, E)$ is defined by

$$F(x) := I_f(x) := I(f \diamond x) := \inf\{\int_S y(s)d\mu(s) : \ y \in L_1(S), \ f \diamond x := f(\cdot, x(\cdot)) \leq y(\cdot) \text{ a.e.}\}.$$

This definition can be reformulated in geometric terms using the linear map

$$A : L_p(S, E) \times L_1(S) \to L_p(S, E) \times \mathbb{R} \tag{1}$$

defined by $A(x, y) := (x, \int_S y(s)ds)$ for $(x, y) \in L_p(S, E) \times L_1(S)$. Here $p \in]1, \infty[$

Lemma 6 *For any integrand f one has*

$$\text{epi}\, I_f = A(L_{p,1}(\text{epi}\, f)).$$

Given $p \in [1, \infty[$, a multimap $M : S \rightrightarrows E$ with nonempty values, we denote by $L_p(M)$ the set of measurable selections of M that are in $L_p(S, E)$:

$$L_p(M) := \{x \in L_p(S, E) : \ x(\cdot) \in M(\cdot) \text{ a.e.}\}.$$

Similarly, if $M : S \rightrightarrows E \times \mathbb{R}$ is a multimap, we denote by $L_{p,1}(M)$ the set of measurable selections of M that are in $L_p(S, E) \times L_1(S)$:

$$L_{p,1}(M) := \{(x, y) \in L_p(S, E) \times L_1(S) : \ (x(\cdot), y(\cdot)) \in M(\cdot) \text{ a.e.}\}.$$

Lemma 7 *Let $f : S \times E \to \overline{\mathbb{R}}$ be a normal integrand such that* $\operatorname{dom} I_f \neq \emptyset$. *Then*

$$\inf\{\int_S f(s, x(s))d\mu(s) : x \in L_p(S, E)\} = \int_S \inf_{e \in E} f(s, e)d\mu(s).$$

Lemma 8 *Let $h : S \times E \to \overline{\mathbb{R}}$ be a normal integrand that is positively homogeneous in its second variable and let $x^* \in L_q(S, E^*)$ such that $\langle x^*, \cdot \rangle \leq I_h(\cdot)$. Then one has $h \diamond 0 = 0$ and $x^* \in L_q(\partial h \diamond 0)$: $x^*(s) \in \partial h(s, \cdot)(0)$ a.e..*

Proof. Since for all $s \in S$ $h_s(0) \in \{0, -\infty\}$ and since $0 = \langle x^*, 0\rangle \leq I_h(0)$ we get $h_s(0) = 0$ a.e.. Thus, any $x \in L_p(S, E)$ we have

$$0 = \inf \int_T (h(s, x(s)) - \langle x^*(s), x(s)\rangle d\mu(s) = \int_S \langle x^*, 1_T x\rangle \leq I_h(1_T x) = \int_T h \diamond x$$

Then, by the preceding lemma in which we take $f(s, e) := h(s, e) - \langle x^*(s), e\rangle$, we get $x^*(s) \leq h(s, \cdot)$ a.e., as announced. $\square$

Integral functionals enjoy some properties of convex functions.

Proposition 9 *(**Bismuth**) If $(S, \mathcal{S}, \mu)$ has no atom and if f is convex in its second variable, I_f is continuous on the whole of $L_p(S, E)$ whenever it is continuous at some point of $L_p(S, E)$.*

Proposition 10 *(**Giner**) If $(S, \mathcal{S}, \mu)$ has no atom and if f is a nonconvex normal integrand, then I_f is Lipschitzian on bounded sets whenever it is Lipschitzian around some point of $L_p(S, E)$.*

Proposition 11 *(**Giner**) If $(S, \mathcal{S}, \mu)$ has no atom and if f is a nonconvex normal integrand, then any local minimizer of I_f is a global minimizer.*

5 Calmness

Let us recall a classical result showing that calmness can be formulated in various equivalent ways.

Lemma 12 *(**Nemeth**) For $F : X \to \overline{\mathbb{R}}$ finite at x in a normed space X the following conditions are equivalent and are satisfied whenever $\partial_D F(x)$ is nonempty:*

(a) F is calm at x: there exist $r \in \mathbb{P} :=]0, \infty[$ and $c \in \mathbb{R}_+$ such that

$$\forall w \in B(x,r) \qquad F(w) - F(x) \geq -c \|w - x\| ;$$

(b) there exists $c \in \mathbb{R}_+$ such that $F^D(x,v) \geq -c\|v\|$ for all $v \in X$;

(c) for all $v \in X$ one has $F^D(x,v) > -\infty$;

(d) one has $F^D(x,0) = 0$.

Proof. The implications (a)⇒(b)⇒(c)⇒(d) are obvious. Suppose (a) does not hold: there exists a sequence (x_n) such that $r_n := \|x_n - x\| \leq n^{-2}$ and $F(x_n) - F(x) \leq -n^2 r_n$. Then, setting $t_n := n r_n$ and $v_n := t_n^{-1}(x_n - x)$ one has $(v_n) \to 0$ and $t_n^{-1}(F(x + t_n v_n) - F(x)) \leq -n$, so that $F^D(x, 0) = -\infty$ and (d) does not hold. □

6 Primal approaches to subdifferentials of integral functionals

6.1 The case of the incident subdifferential

The following result is remarkable: it does not require any restrictive assumption.

Theorem 13 *(**Giner**) Let $f : S \times E \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ be a normal integrand finite at $x \in L_p(S, E)$. Let $h : S \times E \to \overline{\mathbb{R}}$ be the integrand given by $h(s, v) := f_s^I(x(s), v)$ for $(s, v) \in S \times E$. Then, one has $F^I(x, \cdot) \leq I_h(\cdot)$ or*

$$(I_f)^I(x, u) \leq I(f^I \diamond (x, u)) \qquad \forall u \in L_p(S, E), \tag{2}$$

$$\partial_I I_f(x) \subset L_q(\partial_I f \diamond x), \tag{3}$$

with $q := (1 - 1/p)^{-1}$, $L_q(\partial_I f \diamond x) := \{x^* \in L_q(S, E^*) : x^*(s) \in \partial_I f_s(x(s)) a.e.\}$.

Lemma 14 *Given a measurable multimap $M : S \rightrightarrows E \times \mathbb{R}$, $z := (x, y) \in L_{p,1}(M)$, let $T_z^I M$ be the multimap $s \rightrightarrows T^I(M(s), z(s))$. Then one has:*

$$L_{p,1}(T_z^I M) \subset T^I(L_{p,1}(M), z).$$

Proof of Theorem 13. (a) Relation (2) is equivalent to $\operatorname{epi} I_h \subset \operatorname{epi} I_f^I(x,\cdot)$. This inclusion is a consequence of the following relations in which $z(s) := (x(s), f(x(s), s))$, $r := I_f(x)$, and $T^I(\operatorname{epi} f, z)$ denotes the multimap $s \rightrightarrows T^I(\operatorname{epi} f, z(s))$:

$$\begin{aligned}
\operatorname{epi} I_h &= A(L_{p,1}(\operatorname{epi} h)) && \text{(Lemma 6)} \\
&= A(L_{p,1}(T^I(\operatorname{epi} f, z))) && \text{(definition of } h := f^I(x(\cdot),\cdot)) \\
&\subset A(T^I(L_{p,1}(\operatorname{epi} f), z)) && \text{(proposition 14)} \\
&\subset T^I(A(L_{p,1}(\operatorname{epi} f), Az) && \text{(Lemma 4)} \\
&= T^I(\operatorname{epi} I_f, (x, r)) && \text{(Lemma 6).}
\end{aligned}$$

(b) It is a consequence of Lemma 8 and of the definition of $\partial_I I_f$. □

6.2 The case of the circa-subdifferential (Clarke's subdifferential)

A similar approach can be conducted to the Clarke subdifferential, even if the integrand is not locally Lipschitzian. However, we need a growth assumption ensuring the openness property of Lemma 4:

(C) there exist $a \in L_1(S, \mathbb{R})$, $c \in \mathbb{R}_+$ such that

$$f(s,e) \geq a(s) - c\,\|e\|^p \qquad \forall (s,e) \in S \times E.$$

Lemma 15 *Let* $A : L_p(S,E) \times L_1(S) \to L_p(S,E) \times \mathbb{R}$ *be given by* $A(x,y) := (x, \int_S y(s)ds)$ *for* $(x,y) \in L_p(S,E) \times L_1(S)$ *and let* $f : S \times E \to \mathbb{R}_\infty$ *be a normal integrand satisfying condition (C). Then, for all* $x \in L_p(S,E)$ *such that* $f \diamond x \in L_1(S)$*,* A *is open from* $L_{p,1}(\operatorname{epi} f)$ *onto* $\operatorname{epi} I_f$ *at* $(x, f \diamond x)$.

Theorem 16 *(**Giner, 1998**) Let* $f : S \times E \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ *be a normal integrand finite at* $x \in L_p(S,E)$ *and satisfying condition (C). Let* $h : S \times E \to \overline{\mathbb{R}}$ *be the integrand given by* $h(s,v) := f_s^C(x(s),v)$ *for* $(s,v) \in S \times E$*. Then, for all* $u \in L_p(S,E)$ *one has* $(I_f)^C(x,u) \leq I_h(u)$ *or*

$$(I_f)^C(x,u) \leq I(f^C \diamond (x,u)), \tag{4}$$

$$\partial_C I_f(x) \subset L_q(\partial_C f \diamond x). \tag{5}$$

6.3 The case of the directional (=contingent) subdifferential

The study of the directional subdifferential relies on an analysis of calmness.

Lemma 17 *Suppose* $(S, \mathcal{S}, \mu)$ *has no atom and* $f : S \times E \to \overline{\mathbb{R}}$ *is a measurable integrand. Let* $x \in L_p(S, E)$ *be such that* $f \diamond x := f(\cdot, x(\cdot)) \in L_1(S)$. *Then the following assertions are equivalent:*

(a) $F := I_f$ *is calm at* x *on* $X := L_p(S, E)$;

(b) f *satisfies the p-calmness condition:*

(C_p*) there exist* $a \in L_q(S)$, $b \in L_p(S, E)$, $c > 0$ *and a null set* N *such that*

$$\forall (s, e) \in (S \backslash N) \times E \qquad f_s(x(s) + e) - f_s(x(s)) \geq a(s) \|e\| - c \|e\|^p .$$

Theorem 18 *Let* $f : S \times E \to \overline{\mathbb{R}}$ *be a normal integrand such that* I_f *is calm at* $x \in L_p(S, E)$ *such that* $f \diamond x \in L_1(S)$. *Then, for all* $v \in L_p(S, E)$ *one has*

$$I(f^D \diamond (x, v)) \leq F^D(x, v) := I_f^D(x, v),$$
$$L_q(\partial_D f \diamond x) \subset \partial_D I_f(x).$$

7 The case of the firm (=Fréchet) subdifferential

We need a prelimary result in a form slightly more precise than the classical result of Krasnoselskii about Nemytskii operators.

Let us consider the following growth condition in which p, $q \in [1, \infty[$, E, F are Banach spaces and $g : S \times E \to F$ is a measurable map:

(G) there exist $a \in \mathcal{L}_q(S, \mathbb{R})$, $b \in \mathcal{L}_p(S, E)$, $c \in \mathbb{R}_+$ such that

$$\forall_\mu s \in S, \forall e \in E \qquad \|g(s,e)\| \leq a(s) - c\,\|e - b(s)\|^{p/q}.$$

Lemma 19 *If $g : S \times E \to F$ satisfies (G), the following assertions hold:*

(a) for all $u \in \mathcal{L}_p(S, E)$ the map $v := g \diamond u := g(\cdot, u(\cdot))$ belongs to $\mathcal{L}_q(S, F)$;

(b) if for some $u \in \mathcal{L}_p(S, E)$ and all $s \in S$ the map $g_s := g(s, \cdot)$ is continuous at $u(s)$, then the Nemytskii map $G : \mathcal{L}_p(S, E) \to \mathcal{L}_q(S, F)$ given by $G(u) = g(\cdot, u(\cdot))$ is continuous at u.

Theorem 20 *Suppose that for $p \in]1, \infty[$ the integrand f is p-calm at $x \in L_p(S, E)$. Then, for $q := (1 - 1/p)^{-1}$ one has the inclusion*

$$L_q(\partial_F f \diamond x) \subset \partial_F I_f(x).$$

Proof. Let $x^* \in L_q(\partial_F f \diamond x)$ and let $g : S \times E \to \mathbb{R}$ be given by $g(s,0) = 0$,

$$g(s,e) = -\frac{1}{\|e\|}(f(x(s)+e) - f(x(s)) - \langle x^*(s), e\rangle) \qquad (s,e) \in S \times (E\backslash\{0\})$$

g and $g^+ := \max(g,0)$ are measurable and since f is p-calm at x one has

$$g(s,e) \leq a(s) + \|x^*(s)\| + c\,\|e\|^{p-1} \qquad \forall (s,e) \in (S\backslash N) \times E.$$

Since $p-1 = p/q$ and since $x^* \in L_q(S, E^*)$, we see that g satisfies condition (G) with a changed into $a + \|x^*(\cdot)\|$. Proposition 19 and the fact that $g^+(s,e) \to 0$ as $e \to 0$ ensure that $g^+ \diamond v \to 0$ in $L_1(S,\mathbb{R})$ as $v \to 0$ in $L_p(S,E)$. That shows that $x^* \in \partial_F I_f(x)$. □

One may wonder whether one can drop the p-calmness condition in the preceding statement. The following counter-example shows that it is not the case.

Counter-example. Let $S := [0,1]$ with its Lebesgue measure and for $n > 1$ let $S_n := [0, n^{-4}]$. Let us consider the integrand $f : S \times \mathbb{R} \to \mathbb{R}$ given by $f(s,e) := e^3$ and the associated integral functional I_f on $L_2(S,\mathbb{R})$. For all $s \in S$ we have $\partial_F f_s(0) = \{0\}$. Thus, for $x := 0$ we have $L_2(\partial_F f \diamond x) = \{0\}$. However we have $\partial_F I_f(x) = \varnothing$ since for $x_n := -n 1_{S_n}$ we have $\|x_n\|_2 = 1/n$ but $f(x_n)/\|x_n\|_2 = -n$, so that for any $x^* \in L_2(S,\mathbb{R})$ one has $\lim_n (1/\|x_n\|_2)(f(x_n) - f(0) - \langle x^*, x_n\rangle) = -\infty$. □

On the other hand, the following criterion entails p-calmness.

Criterion 21 *If E is an Asplund space and the following condition holds, then f is p-calm at x : there exist $a \in L_q(S, \mathbb{R})$ and $c \in \mathbb{R}_+$ such that for all $(s, e) \in S \times E$, $e^* \in \partial_F f_s(e)$ one has*

$$\|e^*\| \leq a(s) + c \|e\|^{p-1}.$$

7.1 The case of the limiting subdifferential

We just quote the following result.

Theorem 22 *[21] Let f be a normal integrand satisfying condition (C). Suppose $L_p(S,E)$ is separable. Then, for any $x \in L_p(S,E)$ such that $f \in L_1(S)$ one has*

$$L_q(\partial_L f \diamond x) \subset \partial_L F(x).$$

If $L_q(\partial_L f \diamond x) \neq \varnothing$ then one even has

$$L_q(\overline{\mathrm{co}}\partial_L f \diamond x) \subset \partial_L F(x).$$

8 The benefits of regularity

In the sequel we say that the normal integrand f is A-B regular along $x \in L_p(S,E)$ if for some null subset N of S one has $\partial_A f_s(x(s)) = \partial_B f_s(x(s))$ for all $s \in S \backslash N$. We dispose of several results showing that regularity of the integrand f can be transferred to regularity of the integral functional $F := I_f$. We start with the weakest regularity or proto-differentiability (or epi-differentiability).

Theorem 23 *Suppose the normal integrand f is D-I regular along $x \in L_p(S,E)$ and I_f is calm at x. Then*

$$\partial_D F(x) = \partial_I F(x)$$

If I_f is calm at x and if $f^D_\cdot(x(\cdot),\cdot) = f^I_\cdot(x(\cdot),\cdot)$ then

$$F^D(x,u) = F^I(x,u) \qquad \forall u \in L_p(S,E).$$

Proof. The first relation stems from the inclusions $L_q(\partial_D f \diamond x) \subset \partial_D F(x)$ and $\partial_I F(x) \subset L_q(\partial_I f \diamond x)$ since the inclusion $\partial_D F(x) \subset \partial_I F(x)$ is always valid.

The second relation is a consequence of the inequalities $F^D(x,u) \leq F^I(x,u)$ and, under the assumption $f^D_\cdot(x(\cdot),\cdot) = f^I_\cdot(x(\cdot),\cdot)$ that entails $f^D \diamond (x,u) = f^I \diamond (x,u)$

$$F^I(x,u) \leq I(f^I \diamond (x,u)) = I(f^D \diamond (x,u)) \leq F^D(x,u).$$

Note that under the assumption of the theorem, for $\partial = \partial_D$ or $\partial = \partial_I$ we dispose of representations

$$\begin{aligned}\partial F(x) &= L_q(\partial f \diamond x),\\ F^D(x,u) &= I(f^D \diamond (x,u)) = I(f^I \diamond (x,u)) = F^I(x,u).\end{aligned}$$

Theorem 24 *Suppose the normal integrand f is F-I regular along $x \in L_p(S,E)$ and is p-calm along x. Then I_f is F-I regular at x and*

$$\partial_F I_f(x) = L_q(\partial_F f \diamond x) = L_q(\partial_I f \diamond x) = \partial_I I_f(x).$$

Proof. Under the p-calmness assumption, by Theorems 13 and 20 we have

$$\partial_I I_f(x) \subset L_q(\partial_I f \diamond x) \quad \text{and} \quad L_q(\partial_F f \diamond x) \subset \partial_F I_f(x).$$

When f is D-F regular along x i.e. $\partial_I f \diamond x = \partial_F f \diamond x$ we obtain $\partial_I I_f(x) \subset \partial_F I_f(x)$ and since the reverse inclusion is always valid, we get the equalities of the statement. □

Theorem 25 *Suppose the normal integrand f satisfies condition (C) and is C-I regular along $x \in L_p(S,E)$. Then I_f is C-I regular at x and*

$$\partial_C I_f(x) = L_q(\partial_C f \diamond x) = L_q(\partial_I f \diamond x) = \partial_I I_f(x).$$

Moreover, for all $u \in L_p(S,E)$ one has

$$(I_f)^C(x,u) = I(f^C \diamond (x,u)) = I(f^C \diamond (x,u)) = (I_f)^C(x,u)$$

Theorem 26 *Suppose the normal integrand f satisfies condition (C) and is C-F regular along $x \in L_p(S, E)$. Then I_f is C-F regular at x, hence C-D-I-F regular at x and*

$$\partial_C I_f(x) = L_q(\partial_C f \diamond x) = L_q(\partial_F f \diamond x) = \partial_F I_f(x).$$

Proof. Under condition (C) and the p-calmness assumption, by Theorems 16 and 20 we have

$$\partial_C I_f(x) \subset L_q(\partial_C f \diamond x) \quad \text{and} \quad L_q(\partial_F f \diamond x) \subset \partial_F I_f(x).$$

When f is C-F regular along x i.e. $\partial_C f \diamond x = \partial_F f \diamond x$ we obtain $\partial_C I_f(x) \subset \partial_F I_f(x)$ and since the reverse inclusion is always valid, we get the equalities of the statement. □

Theorem 27 *[21] With the assumptions of Theorem (22) suppose that f is C-L regular along x. Then F is C-L regular and*

$$\partial_C F(x) = L_q(\partial_L f \diamond x) = \partial_L F(x).$$

9 Legendre functions and integral functionals

Definition 28 *Given a pairing c between Banach spaces X and X', a function $f : X \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ is a Ekeland function with respect to a subdifferential ∂, in short an Ekeland function, if for any $x_1, x_2 \in X$, $x' \in X'$ satisfying $x' \in \partial f(x_1) \cap \partial f(x_2)$ one has $c(x_1, x') - f(x_1) = c(x_2, x') - f(x_2)$.*

Then, the Ekeland transform of f is the function $f^E : X' \to \mathbb{R}_\infty$ given by $f^E(x') := c(x, x') - f(x)$ for $x \in (\partial f)^{-1}(x')$ for $x' \in \partial f(X)$, $f^E(x') = +\infty$ else.

Definition 29 *The function f is called a Legendre function if f is an Ekeland function, if f^E is an Ekeland function and if $x' \in \partial f(x) \Leftrightarrow x \in \partial f^E(x)$.*

Any convex function is a Legendre function, any quadratic function is a Legendre function and of course, any classical Legendre function is a Legendre function.
Example. (a) If $A : X \to Y$ be a linear, continuous and surjective map, then $f := g \circ A$ is an Ekeland function whenever g is an Ekeland function.
Example. (b) Let $f : X \to \mathbb{R}$, be an Ekeland function such that for all $x \in \mathrm{dom}\partial_F f$, $y \in \partial_F f(x)$ there exists some $r > 0$, $c > 0$ such that

$$d(x, (\partial_F f)^{-1}(z)) \leq c \|y - z\| \qquad \forall z \in \partial_F f(W) \cap B(y, r).$$

Then f is a Legendre function.

Theorem 30 *Let f be a normal integrand such that for a.e. $s \in S$ the function f_s is a Legendre function with respect to the incident subdifferential . Then the integral functional I_f associated with f is a Legendre function with respect to the incident subdifferential and its Legendre transform is the integral functional associated with the Legendre transform of the integrand:*

$$(I_f)^L(x^*) = I_{f^L}(x^*) \qquad \forall x^* \in L_q(S, E^*).$$

Such a result can be seen as an extension of the pioneering studies of convex integral functionals made by R.T. Rockafellar in [R1].

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