

# Regularity Properties of the Hamiltonian in Optimal Control

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Happy Birthday, Terry!



# Outline of the talk

- Regularity properties of the Hamiltonian: why are these useful?
- The Hamiltonian is constant (when data does not depend on time)
- The Hamiltonian is Lipschitz when the data is Lipschitz in time
- These are two examples of a principle: the Hamiltonian inherits the regularity of the data w.r.t. time
- A new example: The Hamiltonian has bounded variation if the data has bounded variation w.r.t. time
- Applications and open questions.

# The Optimal Control Problem

$$(P) \left\{ \begin{array}{l} \text{Minimize } g(x(S), x(T)) \\ \text{over } x(\cdot) \in W^{1,1}([S, T], \mathbb{R}^n) \text{ s.t.} \\ \dot{x}(t) \in F(t, x(t)) \text{ a.e.,} \\ h(t, x(t)) \leq 0, \text{ for all } t \in [S, T] \\ (x(S), x(T)) \in C, \end{array} \right.$$

( $g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $C \subset \mathbb{R}^n \times \mathbb{R}^n$  (closed) and  $F(\cdot, \cdot) : [S, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ .)

## Note:

- 'Differential inclusion' formulation
- State constraint ' $h(x(t)) \leq 0$ '

Take a minimizer  $\bar{x}(\cdot)$ .

# Hypotheses

**(H1):**  $F(., .)$  is closed valued,  $F(., x)$  is  $\mathcal{L}$  - measurable for each  $x \in \mathbb{R}^n$

**(H2):** There exist  $c > 0$  and  $k > 0$  and  $\bar{\delta} > 0$  such that

$$F(t, x) \subset F(t, x') + k(|x - x'|)B \quad \text{and} \quad F(t, x) \in c\mathbb{B} .$$

for all  $x, x' \in \bar{x}(t) + \bar{\delta}B$ ,  $v \in F(t, x)$ , a.e.  $t \in [S, T]$ .

**(H3):**  $h(.)$  is continuously differentiable.

Define Hamiltonian  $H(., ., .) : [S, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$

$$H(t, x, p) := \sup_{v \in F(t, x)} p \cdot v$$

# The Hamiltonian Inclusion

**Theorem (Measurable Time Dependence).** Take a minimizer  $\bar{x}(\cdot)$ . Assume (H1)-(H3).

Then there exist  $p(\cdot) \in W^{1,1}$ ,  $\mu(\cdot) \in NBV^+(S, T)$  and  $\lambda \geq 0$  such that

- (i):  $\text{supp}\{\mu\} \subset \{t \mid h(\bar{x}(t)) = 0\}$
- (ii)  $(p(\cdot), \lambda, \mu(\cdot)) \neq (0, 0, 0)$ ,
- (iii)  $(-\dot{p}(t), \dot{\bar{x}}(t)) \in \text{co } \partial_{x,p} H(t, \bar{x}(t), q(t))$  a.e. ,
- (iv)  $(q(S), -q(T)) \in \lambda \partial g(\bar{x}(S), \bar{x}(T)) + N_C(\bar{x}(S), \bar{x}(T))$ ,

$$\text{where } q(t) = \begin{cases} p(S) & \text{if } t = S \\ p(t) + \int_{[S,t]} \nabla h(\bar{x}(s)) \mu(ds) & \text{if } t \in (S, T] \end{cases}$$

(Gives Pontryagin Max. Principle when  $F(t, x) = f(t, x, U)$ .)

# Regular Time Dependence: Nec. Conditions

- $F(t, x)$  is indep. of  $t \implies H(t, \bar{x}(t), q(t)) = \text{const.}$  on open interval  $(S, T)$
- $F(\cdot, x)$  is Lipschitz  $\implies t \rightarrow H(t, \bar{x}(t), q(t))$  is Lipschitz on open interval  $(S, T)$

Not obvious because

$$H(t, \bar{x}(t), q(t)) = \sup_{v \in F(t, x)} (p(t) + \int_{[S, t]} \nabla h(\bar{x}(t)) \mu(ds)) \cdot v$$

and  $\mu(\cdot)$  may have jumps!

Also:

' $F(t, x)$  is convex for each  $(t, x)$ '

$\implies$  above relations are true on closed interval  $[S, T]$

(Refinement due to Aseev and Arutyunov, '94)

# Idea of Proof

By considering transformation of the independent variable

$$\sigma(s) = \int_{[S,t]} (1 + w(s)) ds, \quad w(s) \in [1 - \epsilon, 1 + \epsilon]$$

show that  $(\bar{x}(s), \bar{z}(s) = s)$  is minimizer for problem with dynamics:

$$(\dot{x}(s), \dot{z}(s)) \in \{((1 + w)v, (1 + w)) \mid v \in F(z(s), x(s)), -\epsilon \leq w \leq \epsilon\}$$

- Richer class of variations (perturb state trajectories also by 'scaling' time variable) yield **extra information**:

$$H(t, \bar{x}(t), q(t)) = r(s) \quad \text{for } t \in (S, T) \quad (1)$$

for some Lipschitz continuous function  $r(\cdot)$  satisfying

$$\dot{r} \in \partial_t H(t, \bar{x}(t), q(t))$$

- additional analysis to extend to (1) to all  $[S, T]$ .
- $F(t, x)$  must be Lipschitz continuous in both variables, because time is now a state variable.



# Regularity of the Hamiltonian: Open Questions

Recall

- $F(t, x)$  is independent of  $t \implies H(t, \bar{x}(t), q(t)) = c$
- $F(\cdot, x)$  is Lipschitz  $\implies t \rightarrow H(t, \bar{x}(t), q(t))$  is Lipschitz

Interpretation:

'The Hamiltonian inherits the time-regularity properties of the dynamics'

Does the Hamiltonian inherit other forms of continuity?

' $t \rightarrow F(\cdot, x)$  is continuous'  $\stackrel{?}{\implies}$  'Hamiltonian is continuous?'

We answer related questions . . .

# Why is Regularity Useful?

- Lagrangian mechanics constancy of Hamiltonian gives **invariants of motion**.

$x(\cdot)$  moves in a conservative force field ( $F(x) = \nabla\phi(x)$ ).  
Motion  $\bar{x}(\cdot)$  renders stationary 'the action':

$$\int \left( \phi(x(t)) - \frac{1}{2}\dot{x}^2(t) \right) dt$$

Hamiltonian is  $\phi(\bar{x}(t)) + \frac{1}{2}\dot{\bar{x}}^2(t)$  (**conservation of energy**)

- Optimality conditions on singular arcs
- Conditions for regularity of optimal controls
- Existence of non-degenerate multipliers

# Functions of Bounded Variation

Classical concept:

$r(\cdot) : [S, T] \rightarrow \mathbb{R}$  has **bounded variation** means

$$\eta(T) < +\infty$$

in which

$$\eta(t) := \sup_{\mathcal{T}(t)} \left\{ \sum_{i=0}^{N-1} |r(t_{i+1}) - r(t_i)| \right\}$$

(Sup taken over all partitions  $\mathcal{T}(t)$  ( $\{t_0 = S, \dots, t_N = t\}$ ) of  $[S, t]$ .)

$\eta(t)$  is called the **cummulative variation function**

- $\eta(\cdot)$  is monotone increasing
- $\eta(\cdot)$  has a countable number of continuity points
- $\eta(\cdot)$  has everywhere left and right limits

# Generalization to Multifunctions

Take a multifunction  $F : [S, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$  and an  $F$ -trajectory  $\bar{x}(\cdot)$ .

Several ways to define ' $t \rightarrow F(t, \cdot)$ ' has bounded variation

**Definition.**  $t \rightarrow F(t, \cdot)$  has **bounded variation** along  $\bar{x}(\cdot)$  if  $\eta(T) < +\infty$ , where

$$\eta(t) := \sup_{\mathcal{T}} \left\{ \sum_{i=0}^{N-1} \sup \{ d_H(F(t_{i+1}, x), F(t_i, x)) \mid x \in G \} \right\} .$$

Take supremum over partitions  $\mathcal{T} = \{t_0 = S, \dots, t_N = t\}$  of  $[S, t]$ .

$$G := \{ \bar{x}(t) \mid t \in [S, T] \}$$

$\eta(\cdot)$  is called the **cummulative variation of  $t \rightarrow F(t, \cdot)$**



# Precedents: Moreau's Sweeping Processes

Take a closed, convex multifunction  $C(\cdot) : [S, T] \rightarrow \mathbb{R}^n$ .  
Sweeping processes are state trajectories for

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) \\ x_0 = x_0 \end{cases}$$

(Moreau, 1973)

Hypotheses:  $\sup_{\mathcal{T}} \left\{ \sum_{i=0}^{N-1} \sup_{v \in C(t_{i+1})} d_{C(t_i)}(v) \right\} < \infty$  .

(Supremum over partitions  $\mathcal{T} = \{t_0 = S, \dots, t_N = t\}$  of  $[S, T]$ .)

(Early example of use of BV multifunctions)

# Properties

Take a multifunction  $F(., .)$ ; of bounded variation along  $\bar{x}(.)$ .

Write  $\eta(.) =$  cumulative variation function. Then

$$d_H(F(t, x'), F(s, x')) \leq \eta(t) - \eta(s)$$

for all  $[s, t] \subset [S, T]$  and  $x' \in x([S, T])$ .

**Multifunctions of bounded variation have many 'classical' properties:**

**(a):** Take any  $s \in [S, T]$  and  $t \in (S, T]$ . The one-sided limits

$$F(s^+, x) := \lim_{s' \downarrow s} F(s', x) \quad \text{and} \quad F(t^-, x) := \lim_{t' \uparrow t} F(t', x)$$

exist for every  $x \in G$ .

**(b):** There exists a countable set  $\mathcal{A}$  such that,

$$\lim_{t' \rightarrow t} d_H(F(t', x), F(t, x)) = 0.$$

for every  $t \in (S, T) \setminus \mathcal{A}$  and  $x \in G$

# Examples of Multifunctions having Bounded Variation

Class of multifunctions  $t \rightarrow F(.,.)$  with bdd. var. (along some  $\bar{x}(.)$ ) is much larger than the class of Lip. multifunctions  $t \rightarrow F(t, .)$ .

Examples of Multifunctions having bounded variation include:

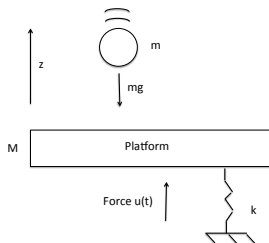
- $F(.,.)$ 's with a finite number of **fractional singularities**, e.g.

$$F(t, x) = \sum_{i=1}^N |t - t_i|^{\frac{1}{2}} \tilde{F}_i(x) \quad (\tilde{F}_i(.) \text{ 'smooth' })$$

- $F(.,.)$ 's with a finite number of **interior discontinuities**
- $F(.,.)$ 's with **end-time discontinuities**

BUT **some Hölder  $t \rightarrow F(t, .)$ 's do not have bounded variation.**

# Example (BV Data)

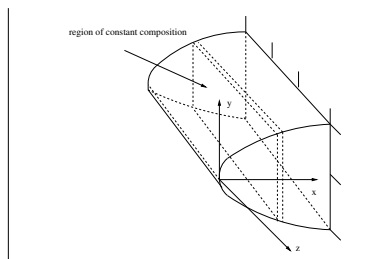


$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} d(t).$$

(Controlled differential equation, in which the control satisfies  $|u(t)| \leq K$ , and which can be reformulated as a differential inclusion with BV time dependence.)



# Example: Optimize bending rigidity in cantilever



- For uniform downward force on leading edge, choose distribution of two materials to maximize bending rigidity  $R$ :

$$R = \text{force}/\text{displacement}$$

- Solve variational problem, in which horizontal displacement  $x$  is time-like variable.
- For rounded leading edge, Lagrangian is non-autonomous with a fractional singularity of at 'time'  $x$ .

$$(L(x, y, u) \sim |x|^\alpha, 0 < \alpha < 1.)$$

# Hamiltonians of Bounded Variation

**Theorem (Palladino + V, 2014).** Take a minimizer  $\bar{x}(\cdot)$ .

Assume

- $F(\cdot, \cdot)$  is convex valued
- $t \rightarrow F(t, \cdot)$  has bounded variation along  $\bar{x}(\cdot)$  with cumulative variation  $\eta(\cdot)$ .

Then the multipliers  $(p(\cdot), \mu(\cdot), \lambda)$  can be chosen to satisfy the following **additional condition**:

- $|H(t, \bar{x}(t), q(t)) - H(s, \bar{x}(s), q(s))| \leq K \times (\eta(t) - \eta(s))$   
for all intervals  $[s, t] \subset [S, T]$ .

i.e. 'Hamiltonian has inherits BV property from data, and has some cumulative variation (modulo scaling)'.

M Palladino and R B Vinter, 'Regularity of the Hamiltonian along Optimal Trajectories', SIAM J. Control and Opt., to appear.

R. B. Vinter, 'Multifunctions of Bounded Variation', submitted

# Idea of Proof

Approximate  $(P)$  by **Autonomous Multistage Problem** on the partition:  
 $\{t_0 = S, \dots, t_N = T\}$ :

$$(P') \left\{ \begin{array}{l} \text{Minimize } g(x(T)) \\ \text{over } x(\cdot) : [S, T] \rightarrow \mathbb{R}^n \text{ s.t.} \\ \dot{x}(t) \in \sum_{i=0}^{N-1} F(t_i, x(t)) \chi_{[t_i, t_{i+1})}(t) \text{ a.e.} \\ \text{and} \\ h(x(t)) \leq 0, \quad \text{for all } t \in [S, T] \\ x(S) = x_0, x(T) \in C. \end{array} \right.$$

Strengthened nec. conditions for multiprocess problem give:

$$|H(t_i, \bar{x}(t_{i+1}), q(t_{i+1})) - H(t_i, \bar{x}(t_i), q(t_i))| \approx 0$$

$\Rightarrow$

$$|H(t_{i+1}, \bar{x}(t_{i+1}), q(t_{i+1})) - H(t_i, \bar{x}(t_i), q(t_i))| \leq K(\eta(t_i) - \eta(t_{i+1})) \dots$$

(desired link between Hamiltonian and cumulative var. fn.!) [\(desired link between Hamiltonian and cumulative var. fn.!\)</a>](#)

# 1st Application

Calculus of Variations:

$$(Q) \begin{cases} \text{Minimize } \int_S^T L(t, x(t), \dot{x}(t)) dt \\ \text{over } x(\cdot) \in W^{1,1}([0, 1]; \mathbb{R}^n) \text{ s.t.} \\ x(S) = x_0 \text{ and, } x(T) = x_1 . \end{cases}$$

(Q) has a minimizer  $\bar{x}(\cdot)$  when:

- (HE):**
- (i):**  $L(\cdot, x, v)$  is  $\mathcal{L} \times \mathcal{B}^{n \times n}$  measurable and  $L(t, \cdot, \cdot)$  is lower semicontinuous for each  $t \in [S, T]$ .
  - (ii):**  $L(t, x, \cdot)$  is convex for each  $(t, x) \in \mathbb{R}^n \times \mathbb{R}^n$ .
  - (iii):** There exists a convex function  $\theta(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  and a number  $\alpha$  such that  $\lim_{r \uparrow \infty} \theta(r)/r = +\infty$ , and  $L(t, x, v) \geq \theta(|v|) - \alpha|x|$  for all  $(t, x, v) \in [S, T] \times \mathbb{R}^n \times \mathbb{R}^n$ .

# Ball Mizel Example - Non Lipschitz Minimzer

$$\begin{cases} \text{Minimize } \int_0^1 \{ r\dot{x}^2(t) + (x^3(t) - t^2)^2 \dot{x}^{14}(t) \} dt \\ \text{over } x \in W^{1,1}([0, 1]; \mathbb{R}) \text{ satisfying} \\ x(0) = 0, \quad x(1) = k. \end{cases}$$

Here,  $r > 0$  and  $k > 0$  are constants, linked by the relationship

$$r = (2k/3)^{12}(1 - k^3)(13k^3 - 7).$$

$\exists \epsilon > 0$  s.t.,  $\forall k \in (1 - \epsilon, 1)$ , the arc  $\bar{x}(t) := kt^{2/3}$  is unique minimizer.

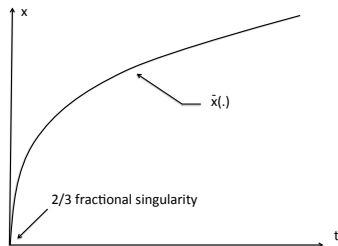


Figure : Non-Lipschitz Minimzer.

# Application 1, Continued

(HE) does not guarantee that  $\bar{x}(\cdot)$  is Lipschitz. But:

**Corollary.** Let  $\bar{x}(\cdot)$  be a minimizer. Assume that

- (HE)
- $t \rightarrow \text{epi } L(t, \cdot, \cdot)$  has bounded variation along  $(\bar{x}(\cdot), \dot{\bar{x}}(\cdot))$

Then  $\bar{x}(\cdot)$  is Lipschitz continuous.

Extends earlier theory:

Replace ' $F(\cdot, x)$  is Lipschitz' by ' $F(\cdot, x)$  has bounded variation'

**Proof Technique:** Use Tonelli Regularity Theory + strengthened conditions . .

## Application 2. Non-degeneracy of Necessary Conditions

If the data is BV then we know

- $|H(t, \bar{x}(t), q(t)) - H(t, \bar{x}(t), q(s))| \leq K \times (\eta(t) - \eta(s))$

for all intervals  $[s, t] \subset [S, T]$  (not just  $(S, T)$ ).

This is an extension to BV time dependence of Arutyunov's **strengthened** necessary conditions.

- The **strengthened** condition can be used to guarantee existence of non-degenerate Lagrange multiplier in some new situations.

Extends earlier theory:

Replace ' $F(\cdot, x)$  is Lipschitz' by ' $F(\cdot, x)$  has bounded variation'

# Linear $L^\infty$ Distance Estimates

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) \\ x(t) \in A \end{cases}$$

$(F(.,.) : [S, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$  and closed set  $A \subset \mathbb{R}^n$ )

A **Linear Distance Estimate** is valid if there exists  $K > 0$  with the property:

For any  $F$ -trajectory  $\hat{x}(\cdot)$  with  $x(S) \in A$ , then there is a feasible  $x(\cdot)$  with  $x(0) = \hat{x}(\cdot)$  and

$$\|x(\cdot) - \hat{x}(\cdot)\|_{L^\infty} \leq K \max_{t \in [S, T]} d_A(\hat{x}(t)).$$

Distance estimates have many uses.



# Linear $L^\infty$ Estimates for BV Data

Theorem (Bettiol, Frankowska, Vinter, JDE 2012).

Assume

- (i): Standard Lipschitz/boundedness conditions
- (ii): 'inward pointing condition':

$$\left( \liminf_{(t',x') \xrightarrow{D} (t,x)} \text{co } F(t', x') \right) \cap \text{int } T_A(x) \neq \emptyset.$$

- (iii):  $t \rightarrow F(t, x)$  is absolutely continuous from the left

Then an  $L^\infty$  Linear Estimate is valid.

(iii) can be replaced by 'F(.,x) has bounded variation'.

# Concluding Remarks

This talk illustrates:

'The Hamiltonian inherits the regularity of  $t \rightarrow F(t, \cdot)$ '

We have seen useful instances of this principle.

$t \rightarrow F(t, \cdot)$  is constant  $\implies$  Hamiltonian is constant

$t \rightarrow F(t, \cdot)$  is Lipschitz  $\implies$  Hamiltonian is Lipschitz

$t \rightarrow F(t, \cdot)$  has bdd. var.  $\implies$  Hamiltonian has bdd. var.  
(new)

Open Question

$t \rightarrow F(t, \cdot)$  is continuous  $\stackrel{?}{\implies}$  Hamiltonian is continuous