Resolvent Averages of Monotone Operators: Dominant and Recessive Properties

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Terryfest 2015
International Conference on Variational Analysis, Optimization and Quantitative Finance
Université de Limoges, France: May 18-22, 2015

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Monotone operators play important roles in optimization and convex analysis. We define a new average of monotone operators by using resolvents. The new average enjoys self-duality and inherits many nice features of given monotone operators. When the monotone operators are positive definite matrices, the new average lies between the harmonic average and arithmetic average. Appropriate limits of resolvent average lead to both harmonic average and arithmetic average. Consequences on matrix functions are also given.
## Outline of Topics

1. Resolvent averages

2. Dominant properties of $\mathcal{R}_\mu(A, \lambda)$
   - At most single-valued or strict monotonicity
   - Uniform monotonicity

3. Recessive properties of $\mathcal{R}_\mu(A, \lambda)$
   - Paramonotonicity and rectangularity
   - Nonexpansive monotonicity and displacement mapping

4. Graphical limits of resolvent averages

5. Extensions and relationships
1. Resolvent averages

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What is a resolvent average?

\( \mathcal{H} \): a real Hilbert space with inner product \( \langle \cdot, \cdot \rangle \).

For a set-valued operator \( A : \mathcal{H} \Rightarrow \mathcal{H} \),

\[
\text{dom } A = \{ x \mid Ax \neq \emptyset \}, \quad \text{ran } A = \bigcup_{x \in \text{dom } A} Ax,
\]

The set-valued inverse \( A^{-1} \) of \( A \):

\[
(y, x) \in \text{gra } A^{-1} \iff (x, y) \in \text{gra } A.
\]

The operator \( A \) is called **monotone** if \( \forall (x_i, x_i^*) \in \text{gra } A, \ i = 1, 2, \)

\[
\langle x_i^* - x_1^*, x_2 - x_1 \rangle \geq 0,
\]

and **strictly monotone** if this inequality is strict whenever \( x_1 \neq x_2 \).
$A : \mathcal{H} \ni \mathcal{H}$ is maximal monotone if the monotone set gra $A$ is not properly contained in any other monotone set.

Id : $\mathcal{H} \to \mathcal{H}$ denotes the identity mapping.

For $\lambda > 0$,

\[ J_A = (\text{Id} + A)^{-1} : \text{resolvent of } A, \]

\[ \lambda A = \lambda^{-1}(\text{Id} - J_{\lambda A}) : \text{Yosida } \lambda\text{-regularization}, \]

\[ S_A = \text{Id} - 2(\text{Id} + A)^{-1} : \text{Rockafellar-Wets regularization of } A. \]
For monotone operators $A_i, i = 1, \ldots, n$ and $\lambda_i > 0$ with $\sum_{i=1}^{n} \lambda_i = 1$, define

$$A = (A_1, A_2, \ldots, A_n),$$

$$\lambda = (\lambda_1, \ldots, \lambda_n).$$

**Resolvent average of** $A_i, i = 1, \ldots, n$

$$\mathcal{R}_\mu(A, \lambda) = [\lambda_1(A_1 + \mu^{-1}\text{Id})^{-1} + \cdots + \lambda_n(A_n + \mu^{-1}\text{Id})^{-1}]^{-1} - \mu^{-1}\text{Id},$$
Question: What does it mean?

\[(\mathcal{R}_\mu(A, \lambda) + \mu^{-1} \text{Id})^{-1} = \lambda_1 (A_1 + \mu^{-1} \text{Id})^{-1} + \cdots + \lambda_n (A_n + \mu^{-1} \text{Id})^{-1},\]

which is equivalent to

\[J_{\mu\mathcal{R}_\mu}(A, \lambda) = \lambda_1 J_{\mu A_1} + \cdots + \lambda_n J_{\mu A_n}.\]
Harmonic average and arithmetic average

Well-known harmonic average and arithmetic average are

\[
\mathcal{H}(A, \lambda) = (\lambda_1 A_1^{-1} + \cdots + \lambda_n A_n^{-1})^{-1},
\]

\[
\mathcal{A}(A, \lambda) = \lambda_1 A_1 + \cdots + \lambda_n A_n.
\]

- If \( \bigcap_{i \in I} \operatorname{ran} A_i = \emptyset \), then \( \mathcal{H}(A, \lambda) \) is empty-valued.
- If \( \bigcap_{i \in I} \operatorname{dom} A_i = \emptyset \), then \( \mathcal{A}(A, \lambda) \) is empty-valued.
Why study resolvent averages?

- \( R_\mu(A, \lambda) \) provides a novel method to obtain set-valued operators from known operators \( A_i \)’s.
- It is very interesting to ask what properties \( R_\mu(A, \lambda) \) has and inherits from those \( A_i \)’s.
- What are the relationships among \( R_\mu(A, \lambda), \mathcal{H}(A, \lambda) \) and \( \mathcal{A}(A, \lambda) \)?
The proximal average of convex functions is defined by

$$\mathcal{P}_\mu(f, \lambda) = \left[ \lambda_1(f_1 + \mu^{-1} j)^* + \cdots + \lambda_n(f_n + \mu^{-1} j)^* \right]^* - \mu^{-1} j,$$

for $f = (f_1, \ldots, f_n)$ with $(\forall i)\ f_i$ being convex functions.

The proximal mapping of $\mathcal{P}_\mu(f, \lambda)$ is the average of proximal mappings of $f_i$'s, namely

$$\text{Prox}_\mu \mathcal{R}_\mu(A, \lambda) = \lambda_1 \text{Prox}_\mu f_1 + \cdots + \lambda_n \text{Prox}_\mu f_n,$$

with the proximal mapping $\text{Prox}_\mu f_i = (\mu \partial f_i + \text{Id})^{-1}$. 
Reformulations of $\mathcal{R}_\mu (A, \lambda)$

**Proposition 1**

We have

$$\mu [\mathcal{R}_\mu (A, \lambda)] = \lambda_1 \mu A_1 + \cdots + \lambda_n \mu A_n.$$  \hspace{1cm} (3)

$$S_{\mu \mathcal{R}_\mu (A, \lambda)} = \lambda_1 S_{\mu A_1} + \cdots + \lambda_n S_{\mu A_n}.$$  \hspace{1cm} (4)

In terms of Yosida $\mu$-regularization of $A_i$'s, we have

**Theorem 2**

$$\mathcal{R}_\mu (A, \lambda) = - \mu [ - (\lambda_1 \mu A_1 + \cdots + \lambda_n \mu A_n)].$$
Proposition 3

$x^* \in \mathcal{R}_\mu(A, \lambda)(x)$ if and only if $(\forall i) \exists x_i \in \text{dom } A_i$ such that

$$\begin{cases} 
    x = \lambda_1 x_1 + \cdots + \lambda_n x_n \\
    x^* \in \bigcap_{i=1}^n (A_i + \mu^{-1} \text{Id})(x_i) - \mu^{-1} x.
\end{cases}$$

Consequently, $\forall x \in \mathbb{R}^N$,

$$\mathcal{R}_\mu(A, \lambda)(x) = \bigcup \left\{ \bigcap_{i=1}^n (A_i + \mu^{-1} \text{Id})(x_i) - \mu^{-1} x : \sum_{i=1}^n \lambda_i x_i = x \right\},$$

$$\text{dom } \mathcal{R}_\mu(A, \lambda) \subset \lambda_1 \text{dom } A_1 + \cdots + \lambda_n \text{dom } A_n.$$
Furthermore, $\forall x \in \mathbb{R}^N, \forall \mu > 0,$

$$\bigcap_{i=1}^{n} A_i(x) \subset \mathcal{R}_\mu(A, \lambda)(x).$$

(8)
Proposition 4

Let \( z^*, z \in \mathbb{R}^N \). Then

\[
\mathcal{R}_\mu((A_1 - z^*, \ldots, A_n - z^*), \lambda) = \mathcal{R}_\mu(A, \lambda) - z^* ,
\]

(9)

\[
\mathcal{R}_\mu((A_1(\cdot - z), \ldots, A_n(\cdot - z)), \lambda) = \mathcal{R}_\mu(A, \lambda)(\cdot - z).
\]

(10)

Proposition 5

Let \( \alpha > 0 \). Then

\[
\mathcal{R}_\mu(\alpha A, \lambda) = \alpha \mathcal{R}_{\alpha \mu}(A, \lambda).
\]

(11)

In particular,

\[
\mathcal{R}_\mu(A, \lambda) = \mu^{-1}\mathcal{R}_1(\mu A, \lambda).
\]

(12)
Example 6

Let $A_i = N_{C_i} \; \forall i$. Then

$$R_\mu(A, \lambda) = \mu^{-1}[(\lambda_1 P_{C_1} + \cdots + \lambda_n P_{C_n})^{-1} - \text{Id}], \quad (13)$$
Fact 7 (Poliquin-Rockafellar'96)

Every mapping $A : \mathcal{H} \rightrightarrows \mathcal{H}$ obeys the identity

$$\text{Id} - (\text{Id} + A)^{-1} = (\text{Id} + A^{-1})^{-1}.$$  \hspace{1cm} (14)

Indeed, the Yosida regularizations of $A$ are related to the resolvents of $A$ by

$$\mu A = (\mu \text{Id} + A^{-1})^{-1} = \mu^{-1}[\text{Id} - (\text{Id} + \mu A)^{-1}] \quad \forall \mu > 0.$$  \hspace{1cm} (15)
Theorem 8 (inverse formula: self duality)

Let $A_i : \mathcal{H} \rightrightarrows \mathcal{H}$ be any set-valued mapping and $\mu > 0$. Assume that $\sum_{i=1}^{n} \lambda_i = 1$ with $\lambda_i > 0$. Then

$$[R_{\mu}(A, \lambda)]^{-1} = R_{\mu^{-1}}(A^{-1}, \lambda), \text{ i.e.,}$$

$$\left[ \left( \lambda_1 (A_1 + \mu^{-1}Id)^{-1} + \cdots + \lambda_n (A_n + \mu^{-1}Id)^{-1} \right)^{-1} - \mu^{-1}Id \right]^{-1} =$$

$$\left( \lambda_1 (A_1^{-1} + \mu Id)^{-1} + \cdots + \lambda_n (A_n^{-1} + \mu Id)^{-1} \right)^{-1} - \mu Id.$$
Who cares about resolvent averages?

Theorem 9 (common solutions to monotone inclusions)

Suppose that for each $i \in I$, $A_i : \mathcal{H} \rightrightarrows \mathcal{H}$ is maximally monotone. Let $x$ and $u$ be points in $\mathcal{H}$. If $\bigcap_{i \in I} A_i(x) \neq \emptyset$, then

$$R_\mu(A, \lambda)(x) = \bigcap_{i \in I} A_i(x). \quad (17)$$

If $\bigcap_{i \in I} A_i^{-1}(u) \neq \emptyset$, then

$$R_\mu(A, \lambda)^{-1}(u) = \bigcap_{i \in I} A_i^{-1}(u). \quad (18)$$
Example 10 (convex feasibility problem)

Let $C_i \subset \mathbb{R}^N$ be non-empty closed convex, and $A_i = N_{C_i}$. If $\bigcap_{i=1}^{n} C_i \neq \emptyset$, then

$$R_{\mu}(A, \lambda)^{-1}(0) = \bigcap_{i=1}^{n} C_i.$$
Who cares about resolvent averages?

Example 11 (homotopy transform)

Let $A_1, A_2 : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ be maximal monotone operators. The mapping $(\forall \lambda \in [0, 1])$ $h_\lambda : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ given by

$$\mathbb{R}^N \ni x \mapsto \begin{cases} A_1x & \text{if } \lambda = 0; \\
A_2x & \text{if } \lambda = 1; \\
\mathcal{R}_1(A, \lambda)x & \text{if } 0 < \lambda < 1. \end{cases}$$

(19)

is a homotopy in the graphical convergence topology. More precisely, $\lambda \mapsto h_\lambda$ is continuous on $[0, 1]$ in the graphical convergence topology.
Resolvent average

Dominant properties

Recessive properties

Graphical limits

Conclusions

Example 1

Define \( A_1(x) = x \) and \( A_2(x) = \begin{cases} 
-1 & \text{if } x < 0 \\
[-1, 1] & \text{if } x = 0 \\
1 & \text{if } x > 0 
\end{cases} \)

Then,

\[
\lambda J_{A_1}(x) = \frac{\lambda}{2} x \quad \text{and} \quad (1 - \lambda) J_{A_2}(x) = \begin{cases} 
(1 - \lambda)(x + 1) & \text{if } x < -1 \\
0 & \text{if } 1 \leq x \leq 1 \\
(1 - \lambda)(x - 1) & \text{if } x > 1 
\end{cases}
\]

Now to see the graph of \( R_1(A, \lambda) \), we use the Minty Parameterization,

\[
\left( J_{\mathcal{R}(A, \lambda)}, x - J_{\mathcal{R}(A, \lambda)} \right)
\]
Example 1 cont.
Example 2

Define $A_1(x) = e^x$ and $A_2(x) = -e^{-x}$. Solving for $J_{A_1}, J_{A_2}$,

$$J_{A_1}(x) = -W(e^x) + x \text{ and } J_{A_2}(x) = W\left(\frac{1}{e^x}\right) + x$$

Where $W$ is the Lambert $W$ function. Now to see the graph of $R_1(A, \lambda)$ we use the Minty Parameterizations,

$$(J_{R(A, \lambda)}, x - J_{R(A, \lambda)})$$
Example 2 cont.
Inheritance of properties

Definition 12

We say that a property \((p)\) is

1. **dominant** if the existence of \(i_0 \in I\) such that \(A_{i_0}\) has property \((p)\) implies that \(R_\mu(A, \lambda)\) has property \((p)\);

2. **recessive** if \((p)\) is not dominant, and for all \(i \in I\), \(A_i\) having property \((p)\) implies that \(R_\mu(A, \lambda)\) has property \((p)\).

3. **indeterminate** if \((p)\) is neither dominant nor recessive.
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5. Extensions and relationships
What have we gained?

**Theorem 13**

Let $A_1, \cdots, A_n : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ be monotone. Then $\mathcal{R}_\mu(A, \lambda)$ is monotone. Moreover,

$$\text{dom } J_{\mu \mathcal{R}_\mu(A, \lambda)} = \text{dom } J_{\mu A_1} \cap \cdots \cap \text{dom } J_{\mu A_n}, \ \text{i.e.,}$$

(20)

Consequently, $\mathcal{R}_\mu(A, \lambda)$ is maximal monotone if and only if $(\forall i) \ A_i$ is maximal monotone.

**Theorem 14 (domain and range of $\mathcal{R}_\mu$)**

Suppose that for each $i \in I$, $A_i : \mathcal{H} \rightrightarrows \mathcal{H}$ is maximally monotone. Then

$$\text{ran } \mathcal{R}_\mu(A, \lambda) \simeq \sum_{i \in I} \lambda_i \text{ ran } A_i, \quad \text{dom } \mathcal{R}_\mu(A, \lambda) \simeq \sum_{i \in I} \lambda_i \text{ dom } A_i.$$  

(21)
Theorem 15 (nonempty interior of the domain, fullness of the domain and surjectivity are dominant)

Suppose that for each $i \in I$, $A_i : \mathcal{H} \rightrightarrows \mathcal{H}$ is maximally monotone.

1. If there exists $i_0 \in I$ such that $\text{int dom } A_{i_0} \neq \emptyset$, then $\text{int dom } R_{\mu}(A, \lambda) \neq \emptyset$.

2. If there exists $i_0 \in I$ such that $\text{dom } A_{i_0} = \mathcal{H}$, then $\text{dom } R_{\mu}(A, \lambda) = \mathcal{H}$.

3. If there exists $i_0 \in I$ such that $A_{i_0}$ is surjective, then $R_{\mu}(A, \lambda)$ is surjective.
Lemma 16

Suppose that for each \( i \in I \), \( N_i : \mathcal{H} \to \mathcal{H} \) is nonexpansive, \( T_i : \mathcal{H} \to \mathcal{H} \) is firmly nonexpansive and set \( N = \sum_{i \in I} \lambda_i N_i \) and \( T = \sum_{i \in I} \lambda_i T_i \). Let \( x \) and \( y \) be points in \( \mathcal{H} \) such that \( \|Tx - Ty\|^2 = \langle x - y, Tx - Ty \rangle \). Then \( T_i x - T_i y = Tx - Ty \) for every \( i \in I \). As a consequence, the following assertions hold:

1. If there exits \( i_0 \in I \) such that \( T_{i_0} \) is injective, then \( T \) is injective.
2. If \( x \) and \( y \) are points in \( \mathcal{H} \) such that \( \|Nx - Ny\| = \|x - y\| \), then \( N_i x - N_i y = Nx - Ny \) for every \( i \in I \).
3. (Reich’ 83) If \( \bigcap_{i \in I} \text{Fix } N_i \neq \emptyset \), then \( \text{Fix } N = \bigcap_{i \in I} \text{Fix } N_i \).

\( T_{i_0} \) being injective: \( T_{i_0}(x) = T_{i_0}y \Rightarrow x = y. \)
Lemma 17

Suppose that for each $i \in I$, $T_i : \mathcal{H} \to \mathcal{H}$ is firmly nonexpansive and set $T = \sum_{i \in I} \lambda_i T_i$. If there exists $i_0 \in I$ such that

$$T_{i_0} x \neq T_{i_0} y \quad \Rightarrow \quad \| T_{i_0} x - T_{i_0} y \|^2 < \langle x - y, T_{i_0} x - T_{i_0} y \rangle,$$

(22)

then $T$ has property (22) as well.
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Lemma 16 gives:

**Theorem 18**

1. Assume that some $A_i$ is at most single-valued. Then $\mathcal{R}_\mu(A, \lambda)$ is also at most single-valued.

2. Assume that some $A_{i_0}$ is strictly monotone. Then $\mathcal{R}_\mu(A, \lambda)$ is also strictly monotone.

- A mapping $A : \mathcal{H} \rightrightarrows \mathcal{H}$ is at most single-valued if for every $x \in \mathcal{H}$, $Ax$ is either empty or a singleton.
- A mapping $A : \mathcal{H} \rightrightarrows \mathcal{H}$ is said to be strictly monotone if whenever $u \in Ax$ and $v \in Ay$ are such that $x \neq y$, then $0 < \langle u - v, x - y \rangle$. 
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Uniform monotonicity and uniform FNE

**Definition 19**

A mapping $A : H \rightrightarrows H$ is monotone with modulus $\phi : [0, \infty] \rightarrow [0, \infty]$ if for every two points $(x, u)$ and $(y, v)$ in $\text{gra} A,$

$$\phi\left(\|x - y\|\right) \leq \langle u - v, x - y \rangle.$$

The mapping $A$ is said to be uniformly monotone with modulus $\phi$ if $\phi(t) = 0 \iff t = 0.$
Definition 20

A mapping $T : \mathcal{H} \to \mathcal{H}$ is firmly nonexpansive with modulus $\phi : [0, \infty] \to [0, \infty]$ if for every pair of points $x$ and $y$ in $\mathcal{H}$,

$$\|Tx - Ty\|^2 + \phi(\|Tx - Ty\|) \leq \langle Tx - Ty, x - y \rangle.$$  

The mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be uniformly firmly nonexpansive with modulus $\phi$ if $\phi(t) = 0 \iff t = 0$.

$\Rightarrow A$ is $\varphi$-monotone $\iff J_A$ is $\varphi$-FNE.
Proposition 21

Suppose that for each $i \in I$, $T_i : \mathcal{H} \to \mathcal{H}$ is firmly nonexpansive with modulus $\phi_i$ which is lower semicontinuous and convex and set $T = \sum_{i \in I} \lambda_i T_i$. Then $T$ is firmly nonexpansive with modulus $\phi = p_\frac{1}{2}(\phi, \lambda)$ which is proper, lower semicontinuous and convex. In particular, if there exists $i_0 \in I$ such that $T_{i_0}$ is $\phi_{i_0}$-uniformly firmly nonexpansive, then $T$ is $\phi$-uniformly firmly nonexpansive.

This gives

Theorem 22 (uniform monotonicity is dominant)

Suppose that for each $i \in I$, $A_i : \mathcal{H} \rightrightarrows \mathcal{H}$ is maximally monotone with modulus $\phi_i$ which is lower semicontinuous and convex. Then $R_\mu(A, \lambda)$ is monotone with modulus $\phi = p_\frac{\mu}{2}(\phi, \lambda)$ which is lower semicontinuous and convex. In particular, if there exists $i_0 \in I$ such that $A_{i_0}$ is $\phi_{i_0}$-uniformly monotone, then $R_\mu(A, \lambda)$ is $\phi$-uniformly monotone.
Definition 23

A mapping $A : \mathcal{H} \rightrightarrows \mathcal{H}$ is $\epsilon$-monotone, where $\epsilon \geq 0$, if $A - \epsilon \text{Id}$ is monotone, that is, if for any two points $(x, u)$ and $(y, v)$ in $\text{gra} \ A$,

$$\epsilon \|x - y\|^2 \leq \langle v - u, x - y \rangle.$$ 

We also say that $A^{-1}$ is $\epsilon$-cocoercive.

Suppose that for each $i \in I$, $0 \leq \alpha_i \leq \infty$ and set $\alpha = (\alpha_1 \cdots, \alpha_n)$. Define

$$r_\mu(\alpha, \lambda) = \left[ \sum_{i \in I} \lambda_i (\alpha_i + \mu^{-1})^{-1} \right]^{-1} - \mu^{-1} \quad \text{and} \quad r(\alpha, \lambda) = r_1(\alpha, \lambda). \quad (23)$$
Theorem 24 (strong monotonicity is dominant)

Suppose that for each $i \in I$, $\epsilon_i \geq 0$, $A_i : \mathcal{H} \rightrightarrows \mathcal{H}$ is maximally monotone and $\epsilon_i$-monotone. Then $\mathcal{R}_\mu(A, \lambda)$ is $\epsilon$-monotone where $\epsilon = r_\mu(\epsilon, \lambda)$. In particular, if there exists $i_0 \in I$, such that $A_{i_0}$ is $\epsilon_{i_0}$-strongly monotone, then $\mathcal{R}_\mu(A, \lambda)$ is $\epsilon$-strongly monotone.

Corollary 25 (cocoerciveness is dominant)

Suppose that for each $i \in I$, $\epsilon_i \geq 0$, $A_i : \mathcal{H} \rightrightarrows \mathcal{H}$ is maximally monotone and $A_i^{-1}$ is $\epsilon_i$-monotone. Then $(\mathcal{R}_\mu(A, \lambda))^{-1}$ is $\epsilon$-monotone where $\epsilon = r_{\mu^{-1}}(\epsilon, \lambda)$. In particular, if there exists $i_0 \in I$ such that $A_{i_0}$ is $\epsilon_{i_0}$-cocoercive, then $\mathcal{R}_1(A, \lambda)$ is $\epsilon$-cocoercive.
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Example 26

Let \( f = \| \cdot \| \), \( A_1 = \partial f \) and \( A_2 = 0 \). Then

\[
J_{A_1} x = \begin{cases} 
(1 - \frac{1}{\|x\|}) x, & \text{if } \|x\| > 1; \\
0, & \text{if } \|x\| \leq 1
\end{cases}
\]

and \( J_{A_2} = \text{Id} \). \( J_{A_1} \) is not an affine relation and \( J_{A_2} \) is linear. However, for \( 0 < \lambda < 1 \), \( \lambda_1 = \lambda \),

\[
J_{R_1(A, \lambda)} x = \lambda J_{A_1} x + (1 - \lambda) J_{A_2} x = \begin{cases} 
\left(1 - \lambda \frac{1}{\|x\|}\right) x, & \text{if } \|x\| > 1; \\
(1 - \lambda) x, & \text{if } \|x\| \leq 1
\end{cases}
\]

is not an affine relation. Thus, \( R_1(A, \lambda) \) is not an affine relation.

Example 26 says: linearity and affinity are not dominant properties w.r.t. \( R_\mu(A, \lambda) \).
Theorem 27 (Linearity and affinity are recessive)

Suppose that for each \( i \in I \), \( A_i : \mathcal{H} \rightrightarrows \mathcal{H} \) is a maximally monotone linear (resp. affine) relation. Then \( R_\mu(A, \lambda) \) is a maximally monotone linear (resp. affine) relation.
Definition 28 (rectangular and paramonotone mappings)

The monotone mapping $A : \mathcal{H} \Rightarrow \mathcal{H}$ is said to be

1. **rectangular** (also known as $3^\ast$ monotone) if for every $x \in \text{dom } A$ and every $v \in \text{ran } A$ we have

\[
\inf_{(z,w) \in \text{gra } A} \left\langle v - w, x - z \right\rangle > -\infty ,
\] 

(24) equivalently, if

\[
\text{dom } A \times \text{ran } A \subseteq \text{dom } F_A .
\] 

(25)

2. **paramonotone** if whenever we have a pair of points $(x, v)$ and $(y, u)$ in gra $A$ such that $\langle x - y, v - u \rangle = 0$, then $(x, u)$ and $(y, v)$ are also in gra $A$. 

Recall

Definition 29 (Fitzpatrick function)

With the mapping $A : \mathcal{H} \rightrightarrows \mathcal{H}$ we associate the Fitzpatrick function $F_A : \mathcal{H} \times \mathcal{H} \to ]-\infty, +\infty]$, defined by

$$F_A(x, v) = \sup_{(z,w) \in \text{gra} A} \left( \langle w, x \rangle + \langle v, z \rangle - \langle w, z \rangle \right), \quad (x, v) \in \mathcal{H} \times \mathcal{H}. \quad (26)$$
Fact 30

Let $A \in \mathbb{R}^{N \times N}$ be monotone and set $A_+ = \frac{1}{2}A + \frac{1}{2}A^\top$. Then the following assertions are equivalent:

1. $A$ is paramonotone;
2. $A$ is rectangular;
3. $\text{rank } A = \text{rank } A_+$;
4. $\text{ran } A = \text{ran } A_+$. 
Example 31

In \( \mathbb{R}^2 \), let \( A_1 = N_{\mathbb{R} \times \{0\}} \). Then \( J_{A_1} \) is the projection on \( \mathbb{R} \times \{0\} \). Since \( A_1 \) is a subdifferential, it is rectangular and paramonotone. Let \( A_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be the counterclockwise rotation by \( \pi/2 \). Then

\[
J_{A_1} = P_{\mathbb{R} \times \{0\}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad J_{A_2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.
\]

Since

\[
A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A_2^+ = \frac{1}{2}(A_2 + A_2^T) = 0,
\]

\( A_2 \) is neither rectangular nor paramonotone.
Letting \( \lambda_1 = \lambda_2 = \frac{1}{2} \), we obtain

\[
\mathcal{R}(A) = \left( \frac{1}{2} J_{A_1} + \frac{1}{2} J_{A_2} \right)^{-1} - \text{Id} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}
\]

and

\[
\mathcal{R}(A)_+ = \frac{1}{2} (\mathcal{R}(A) + \mathcal{R}(A)^\top) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.
\]

By employing Fact 30 we see that \( \mathcal{R}_1(A, \lambda) \) is neither rectangular nor paramonotone.
Outline

1. Resolvent averages

2. Dominant properties of $R_\mu(A, \lambda)$
   - At most single-valued or strict monotonicity
   - Uniform monotonicity

3. Recessive properties of $R_\mu(A, \lambda)$
   - Paramonotonicity and rectangularity
   - Nonexpansive monotonicity and displacement mapping

4. Graphical limits of resolvent averages

5. Extensions and relationships
Lemma 32 (Fitzpatrick function of $R_\mu$)

Suppose that for each $i \in I$, $A_i : \mathcal{H} \rightrightarrows \mathcal{H}$ is maximally monotone. Then

$$F_{\mu R_\mu}(A,\lambda) \leq \sum_{i \in I} \lambda_i F_{\mu A_i} \quad \text{in particular,} \quad F_{R_1}(A,\lambda) \leq \sum_{i \in I} \lambda_i F_{A_i} \quad (27)$$

and

$$\sum_{i \in I} \lambda_i \text{dom } F_{\mu A_i} \subseteq \text{dom } F_{\mu R_\mu}(A,\lambda) \quad \text{in particular,} \quad \sum_{i \in I} \lambda_i \text{dom } F_{A_i} \subseteq \text{dom } F_{R_1}(A,\lambda). \quad (28)$$

Theorem 33 (rectangularity is recessive)

Suppose that for each $i \in I$, $A_i : \mathcal{H} \rightrightarrows \mathcal{H}$ is rectangular and maximally monotone. Then $R_\mu(A, \lambda)$ is rectangular.
To study paramonotonicity, we need:

**Proposition 34**

Suppose that for each $i \in I$, $T_i : \mathcal{H} \to \mathcal{H}$ is firmly nonexpansive and set $T = \sum_{i \in I} \lambda_i T_i$. Then:

1. If for each $i \in I$, given points $x$ and $y$ in $\mathcal{H},$

   $\|T_i x - T_i y\|^2 = \langle x - y, T_i x - T_i y \rangle \Rightarrow$

   \[
   \begin{cases}
   T_i x = T_i (T_i x + y - T_i y) \\
   T_i y = T_i (T_i y + x - T_i x),
   \end{cases}
   \]

   then $T$ also has property (29).

2. If there exists $i_0 \in I$ such that $T_{i_0}$ has property (29) and is injective, then $T$ has property (29) and is injective.
Theorem 35 (paramonotonicity is recessive)

Suppose that for each \( i \in I \), \( A_i : \mathcal{H} \rightrightarrows \mathcal{H} \) is maximally monotone and paramonotone. Then \( R_\mu(A, \lambda) \) is paramonotone.
Outline

1. Resolvent averages
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Definition 36
The mapping $D : \mathcal{H} \to \mathcal{H}$ is said to be a displacement mapping if there exists a nonexpansive mapping $N : \mathcal{H} \to \mathcal{H}$ such that $D = \text{Id} - N$, in which case $D$ is maximally monotone.

Lemma 37
The maximally monotone mapping $A : \mathcal{H} \rightrightarrows \mathcal{H}$ is $\frac{1}{2}$-strongly monotone if and only if $A^{-1}$ is a displacement mapping.

Theorem 38 (being a displacement mapping is recessive)
Suppose that for each $i \in I$, $A_i : \mathcal{H} \rightrightarrows \mathcal{H}$ is a displacement mapping. Then $R_1(A, \lambda)$ is a displacement mapping.
Lemma 39

The maximally monotone mapping $N : \mathcal{H} \rightarrow \mathcal{H}$ is nonexpansive if and only if $N = 2J_B - \text{Id}$ for a maximally monotone and nonexpansive mapping $B$.

Theorem 40 (nonexpansiveness is recessive)

Suppose that for each $i \in I$, $A_i : \mathcal{H} \rightarrow \mathcal{H}$ is a nonexpansive and monotone mapping. Then $\mathcal{R}_1(A, \lambda)$ is nonexpansive. Furthermore, for each $i \in I$, $A_i = 2J_{B_i} - \text{Id}$ where $B_i$ is maximally monotone, nonexpansive and $\mathcal{R}_1(A, \lambda) = 2J_B - \text{Id}$ where $B$ is the maximally monotone and nonexpansive mapping given by $B = \sum_{i \in I} \lambda_i B_i$. 
Theorem 41 (within the class of nonexpansive mappings, being a Banach contraction is dominant)

Suppose that for each $i \in I$, $A_i : \mathcal{H} \to \mathcal{H}$ is nonexpansive and monotone. If there exists $i_0 \in I$ such that $A_{i_0}$ is a Banach contraction, then $R_1(A, \lambda)$ is a Banach contraction.
Theorem 42 (in $MLR(\mathcal{H})$, being $BML(\mathcal{H})$ and being $BMLI(\mathcal{H})$ are dominant)

Suppose that for each $i \in I$, $A \in MLR(\mathcal{H})$ and there exists $i_0 \in I$ such that $A_{i_0} \in BML(\mathcal{H})$. Then $\mathcal{R}_\mu(A, \lambda) \in BML(\mathcal{H})$. Furthermore:

1. If $A_{i_0} \in BMLI(\mathcal{H})$, then $\mathcal{R}_\mu(A, \lambda) \in BMLI(\mathcal{H})$;
2. If $A_{i_0}$ is paramonotone, then $\mathcal{R}_\mu(A, \lambda)$ is paramonotone.

$MLR(\mathcal{H})$: maximally monotone and linear relation on $\mathcal{H}$.
$BML(\mathcal{H})$: bounded monotone linear operators.
$BMLI(\mathcal{H})$: bounded monotone linear operators with a bounded inverse.
Indeterminate properties

1. Being a projection is indeterminate;
2. Being a normal cone operator is indeterminate.
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4 **Graphical limits of resolvent averages**

5 Extensions and relationships
The graphical limits by Attouch, Rockafellar and Wets are effective for analyzing the convergence of sequences of resolvent averages.

**Definition 43**

For a sequence of mappings $S_k : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$, we say that $S_k$ converges graphically to $S$, denoted by $S_k \xrightarrow{g} S$, if

$$\limsup_k (\text{gra } S_k) = \liminf_k (\text{gra } S_k) = \text{gra } S.$$ 

Equivalently, $\forall x \in \mathbb{R}^N$ one has

$$\bigcup_{x_k \to x} \limsup_k S_k(x_k) \subset S(x) \subset \bigcup_{x_k \to x} \liminf_k S(x_k).$$
Fact 44 (Rockafellar & Wets)

(i) If a sequence of maximal monotone mappings $S_k : \mathbb{R}^N \rightharpoonup \mathbb{R}^N$ converges graphically, then the limit mapping $S$ must be maximal monotone.

(ii) For maximal monotone mappings $S_k$ and $S$, for any choice of $\mu > 0$ one has

$$S_k \xrightarrow{g} S \iff (\text{Id} + \mu S_k)^{-1} \xrightarrow{p} (\text{Id} + \mu S)^{-1}.$$
Theorem 45

Let \((A_i,k)_{k \in \mathbb{N}}\) be sequences of maximal monotone mappings and let \((\lambda_i,k)_{k \in \mathbb{N}}\) and \((\mu_k)_{k \in \mathbb{N}}\) be sequences in \((0, +\infty)\) such that that \((\forall i)\) \(A_i,k \xrightarrow{g} A_i, \lambda_i,k \rightarrow \lambda_i > 0\) and \(\mu_k \rightarrow \mu > 0\). Then

\[
\mathcal{R}_{\mu_k}((A_{1,k}, \ldots, A_{n,k}), (\lambda_{1,k}, \ldots, \lambda_{n,k})) \xrightarrow{g} \mathcal{R}_{\mu}(A, \lambda), \text{ as } k \rightarrow \infty.
\]

Moreover, \(\mathcal{R}_{\mu}(A, \lambda)\) is maximal monotone.

Question: What happens for \(\mu \downarrow 0\) or \(\mu \uparrow \infty\)?
Lemma 46

Let \((A_n)_{n \in \mathbb{N}}, A\) be linear operators from \(\mathbb{R}^N\) to \(\mathbb{R}^N\). If \(A_n \xrightarrow{g} A\), then there exists \(M > 0\) such that

\[
\|A_n\| < M \quad \forall n \in \mathbb{N}, \quad \text{and} \quad \|A\| < M.
\]

Consequently, for linear operators \((A_n)_{n \in \mathbb{N}}\) and \(A\) on \(\mathbb{R}^N\), the followings are equivalent:

(i) graphical convergence: \(A_n \xrightarrow{g} A\);
(ii) point-wise convergence: \(A_n \xrightarrow{p} A\);
(iii) norm convergence: \(A_n \xrightarrow{n} A\).
Theorem 47

Let $A_i, i = 1, \ldots, k$ be invertible linear monotone operators. Assume that at least one of $A_i^{-1}, i = 1, \ldots, k$ is strictly monotone. Then

$$\mathcal{R}_\mu(A, \lambda) \to (\lambda_1 A_1^{-1} + \cdots + \lambda_n A_n^{-1})^{-1},$$

when $\mu \to \infty$.

Theorem 48

Let $A_i, i = 1, \ldots, n$ be linear and monotone operators. Then

$$\mathcal{R}_\mu(A, \lambda) \to \lambda_1 A_1 + \cdots + \lambda_n A_n,$$

when $\mu \downarrow 0$. 
When \( A_i = \partial f_i \): 

**Theorem 49**

Let \( f_i : X \to \mathbb{R} \), \( i = 1, \ldots, n \) be proper lower semi-continuous convex functions. We have

(i) When \( \mu \downarrow 0 \),

\[
R_\mu(\partial f, \lambda) \xrightarrow{\text{g}} \partial(\lambda_1 f_1 + \cdots + \lambda_n f_n).
\]

If, in addition, \( \cap_{i=1}^n \text{ri(dom } f_i) \neq \emptyset \), then

\[
R_\mu(\partial f, \lambda) \xrightarrow{\text{g}} \lambda_1 \partial f_1 + \cdots + \lambda_n \partial f_n \quad \text{when } \mu \downarrow 0;
\]
(ii) When $\mu \uparrow \infty$,

$$\mathcal{R}_\mu(\partial f, \lambda) \xrightarrow{g} \partial(\lambda_1 f_1^* + \cdots + \lambda_n f_n^*)^*.$$ 

If, in addition, $\bigcap_{i=1}^n \text{ri}(\text{dom } f_i^*) \neq \emptyset$, then

$$\mathcal{R}_\mu(\partial f, \lambda) \xrightarrow{g} [\lambda_1(\partial f_1)^{-1} + \cdots + \lambda_n(\partial f_n)^{-1}]^{-1} \quad \text{when } \mu \uparrow \infty.$$
When \((\forall i)\) \(A_i\) are positive definite matrices:

Let \(S^N_+\) (resp. \(S^N_{++}\)) be the set of positive semidefinite matrices (resp. positive definite matrices). For symmetric matrices \(X, Y\), if \(X - Y \in S^N_+\) we write \(X \preceq Y\).

**Theorem 50**

Let \(A_1, \ldots, A_n \in S^n_{++}\). We have

1. \(\mathcal{H}(A, \lambda) \preceq \mathcal{R}_\mu(A, \lambda) \preceq A(A, \lambda);\) (31)

2. \(\mathcal{R}_\mu(A, \lambda) \to A(A, \lambda)\) when \(\mu \to 0;\)

3. \(\mathcal{R}_\mu(A, \lambda) \to \mathcal{H}(A, \lambda)\) when \(\mu \to \infty.\)
Corollary 51

Assume that $(\forall i) A_i \in S_{++}^N$ and $\sum_{i=1}^{n} \lambda_i = 1$ with $\lambda_i > 0$. Then

$$(\lambda_1 A_1 + \cdots + \lambda_n A_n)^{-1} \preceq \lambda_1 A_1^{-1} + \cdots + \lambda_n A_n^{-1}.$$ 

Consequently, the matrix function $X \mapsto X^{-1}$ is matrix convex on $S_{++}^N$.

Corollary 52

For every $\mu > 0$, the resolvent average matrix function $A \mapsto R_\mu(A, \lambda)$ given by

$$(A_1, \cdots, A_n) \mapsto [\lambda_1 (A_1^{-1} + \mu^{-1} \text{Id})^{-1} + \cdots + \lambda_n (A_n^{-1} + \mu^{-1} \text{Id})^{-1}]^{-1} - \mu^{-1} \text{Id}$$

is matrix concave on $S_{++}^N \times \cdots \times S_{++}^N$. 


For each $\lambda = (\lambda_1, \cdots, \lambda_n)$ with $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_i > 0 \ \forall i$, the harmonic average matrix function

$$(A_1, \cdots, A_n) \mapsto (\lambda_1 A_1^{-1} + \cdots + \lambda_n A_n^{-1})^{-1}$$

is matrix concave on $S_{++}^N \times \cdots \times S_{++}^N$. Consequently, the harmonic average function

$$(x_1, \cdots, x_n) \mapsto \frac{1}{x_1^{-1} + \cdots + x_n^{-1}}$$

is concave (32) on $\mathbb{R}_{++} \times \cdots \times \mathbb{R}_{++}$. 
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4 Graphical limits of resolvent averages

5 Extensions and relationships
How far does this take us?

- Extensions to averaged mappings by Combettes?
- Relationships to geometric averages of matrices, variational sums of monotone operators by Attouch, Baillon & Thera?
- For general monotone operators, under what conditions
  \[
  \mathcal{R}_\mu(A, \lambda) \xrightarrow{\mathcal{g}} \mathcal{H}(A, \lambda) \text{ when } \mu \uparrow \infty,
  \]
  \[
  \mathcal{R}_\mu(A, \lambda) \xrightarrow{\mathcal{g}} \mathcal{A}(A, \lambda) \text{ when } \mu \downarrow 0
  \]
Thank You Very Much!
Bibliography


