Essential velocities in stratified control systems

Peter R. Wolenski

Louisiana State University

Terry Fest, Limoges, France
May 18-22, 2015

Outline

1. Snell's Law, Bernoulli's Brachistochrone, and Elvis

Outline

1. Snell's Law, Bernoulli's Brachistochrone, and Elvis
2. Structured discontinuous systems
   - Stratified domains
   - Stratified dynamics

Outline

Outline

Outline

Fermat's Principle and Snell's Law

A light beam is directed across two homogeneous and contiguous media $\mathcal{M}_1$ and $\mathcal{M}_2$. We observe the light is refracted, but **how**?

What is the path traversed by the beam?

Fermat's Principle says whatever the medium, the beam takes the path of least time among all possible paths.

Suppose the light beam travels with speed s_1 through $\mathcal{M}_1$ and with speed s_2 through $\mathcal{M}_2$, and where the beam originates from some point $P_1 \in \mathcal{M}_1$ with an incidental angle of θ_1.

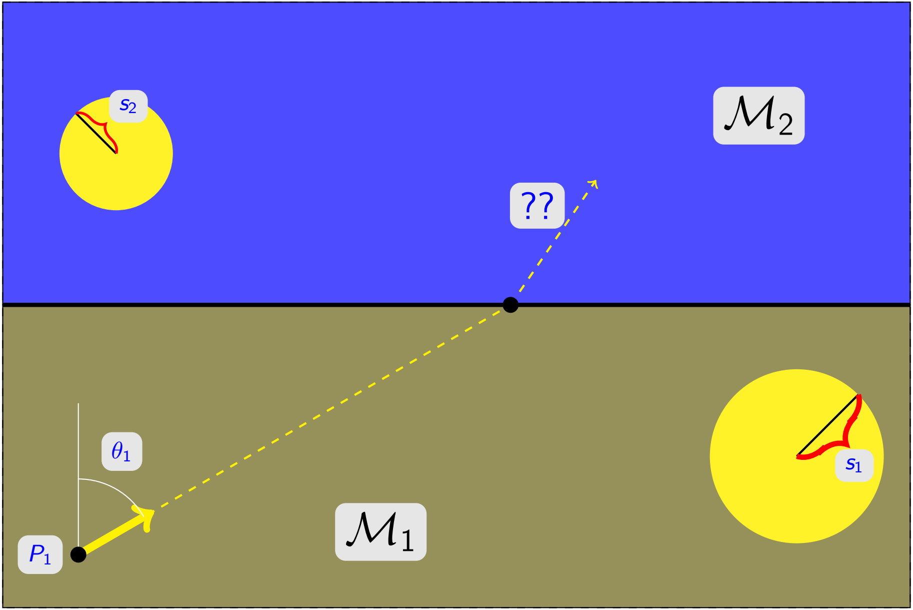
Fermat's Principle and Snell's Law

A light beam is directed across two homogeneous and contiguous media $\mathcal{M}_1$ and $\mathcal{M}_2$. We observe the light is refracted, but **how**?

What is the path traversed by the beam?

Fermat's Principle says whatever the medium, the beam takes the path of least time among all possible paths.

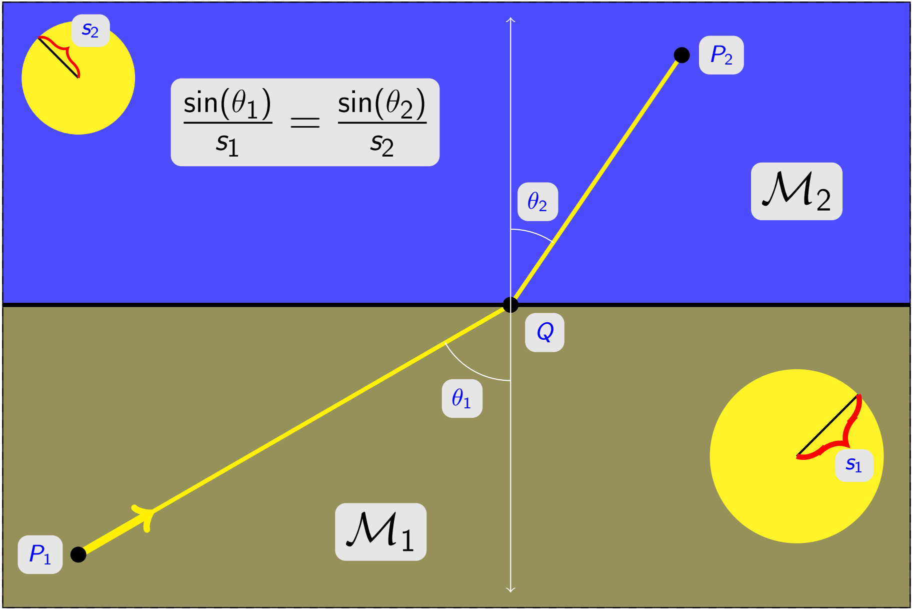
Suppose the light beam travels with speed s_1 through $\mathcal{M}_1$ and with speed s_2 through $\mathcal{M}_2$, and where the beam originates from some point $P_1 \in \mathcal{M}_1$ with an incidental angle of θ_1.


s_2
$\mathcal{M}_2$
??
θ_1
s_1
P_1
$\mathcal{M}_1$

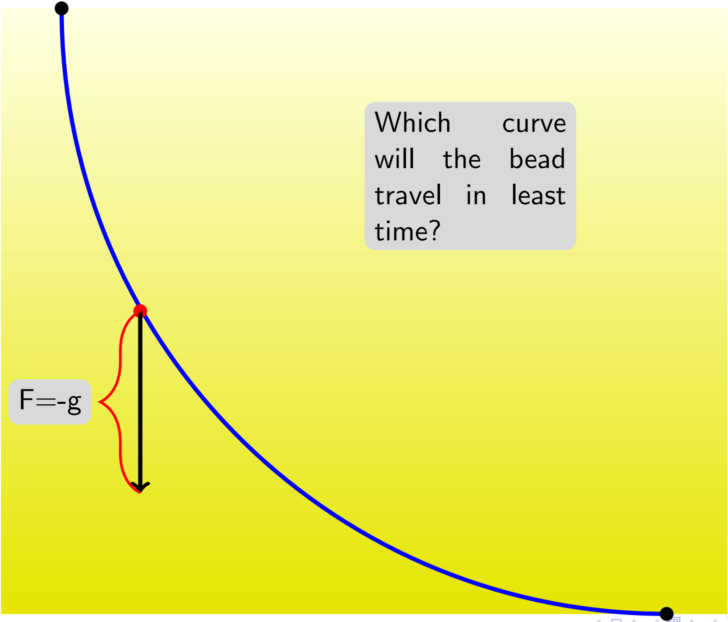
Suppose the beam originates from $P_1 \in \mathcal{M}_1$ with an incidental angle θ_1 (it moves linearly while in one of the mediums) and goes into the second medium through a point $P_2 \in \mathcal{M}_2$. Then among all possible piecewise linear paths that connect P_1 to P_2, the beam travels the path with the least amount of time. This is **Fermat's Principle**.

Let Q denote the point where the beam hits the interface. The time traveled from P_1 to Q is $T_1 := \frac{\|P_1 - Q\|}{s_1}$ and from Q to P_2 is $T_2 := \frac{\|P_2 - Q\|}{s_2}$. The total time traveled is thus $T_1 + T_2$, and minimizing this quantity over all possible Q produces **Snell's Law**:

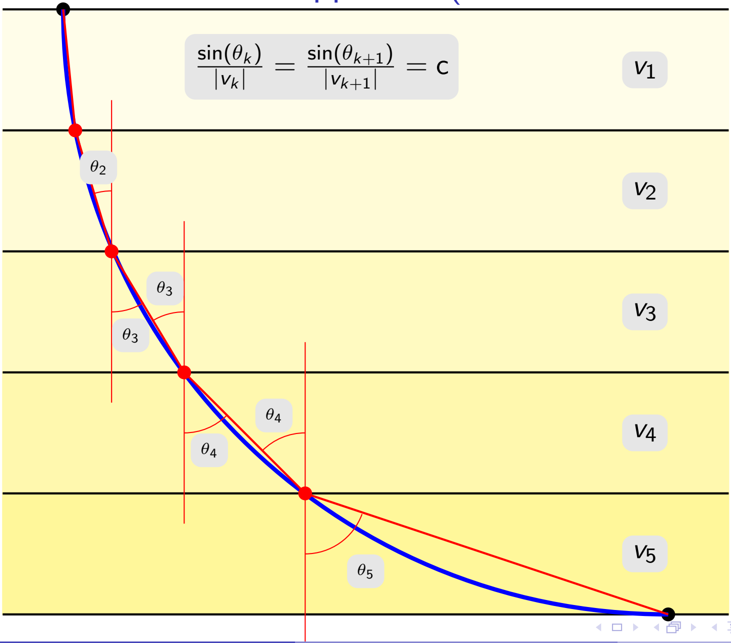
$$\boxed{\frac{\sin(\theta_1)}{s_1} = \frac{\sin(\theta_2)}{s_2}}$$


s_2
$$\frac{\sin(\theta_1)}{s_1} = \frac{\sin(\theta_2)}{s_2}$$
P_2
$\mathcal{M}_2$
θ_2
Q
θ_1
s_1
P_1
$\mathcal{M}_1$

The Brachistochrone



Johann Bernoulli's approach (local constant velocities)

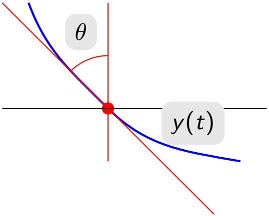


The condition

$$\boxed{\frac{\sin(\theta_1)}{|v_1|} = \frac{\sin(\theta_2)}{|v_2|} = \cdots = \frac{\sin(\theta_N)}{|v_N|} = \mathbf{c}}$$

"limits" to

$$|v| \cdot c = \sin(\theta) = \frac{1}{\sqrt{1+(y')^2}} \quad \Rightarrow \quad t = c \int \frac{|v|}{\sqrt{1 - c^2 |v|^2}}\, dy$$



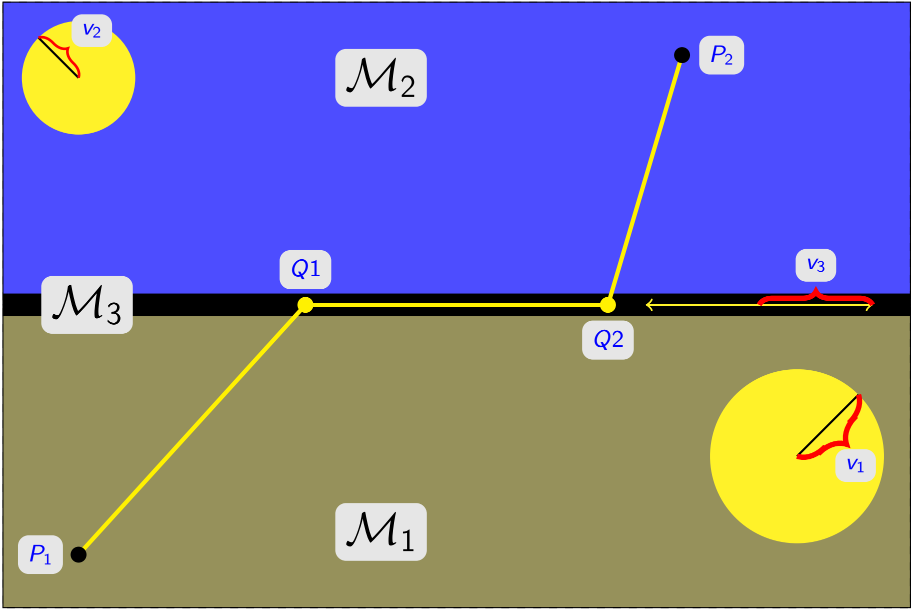
Brachistochrone has $|v(y)| = \sqrt{2\,g\,(y - y_0)}$.

Elvis

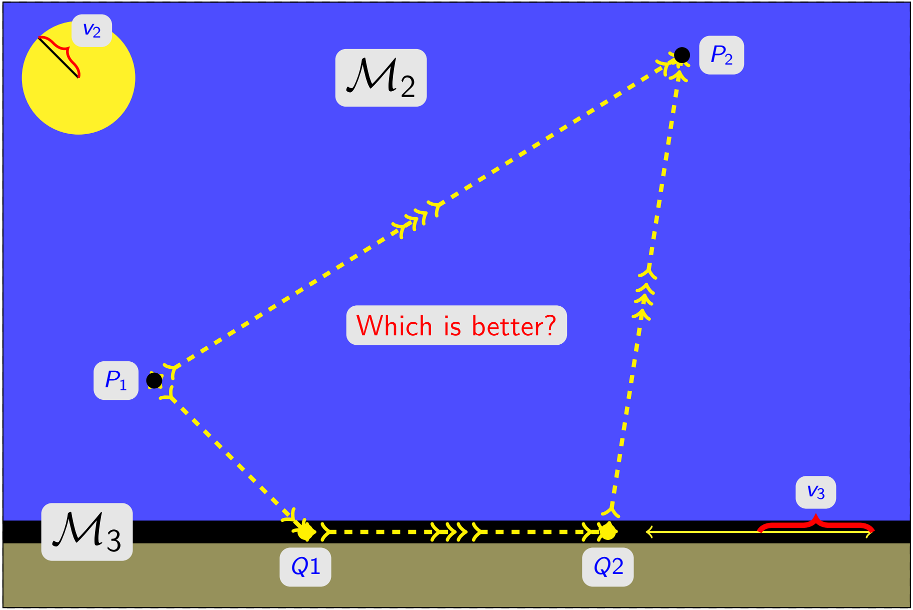
From "Calculating Dogs" by Ivars Peterson, web edition.

In 2003, mathematician Tim Pennings of Hope College in Holland, Mich., revealed to the world that his Welsh corgi, Elvis, appears to be solving a calculus problem when finding the optimal path to fetch a ball. In this case, optimal path means minimizing travel time.

When Elvis and Pennings go to the beach, they always play fetch. Standing at the water's edge, Pennings throws a tennis ball out into the waves, and Elvis eagerly retrieves it. When Pennings throws the ball at an angle to the shoreline, Elvis has several options. He can run along the beach until he is directly opposite the ball, then swim out to get it. Or he can plunge into the water right away and swim all the way to the ball. What happens most the time, however, is that Elvis runs part of the way along the beach, then swims out to the ball.



v_2
$\mathcal{M}_2$
P_2
$Q1$
v_3
$\mathcal{M}_3$
$Q2$
v_1
$\mathcal{M}_1$
P_1


v_2
$\mathcal{M}_2$
P_2
Which is better?
P_1
v_3
$\mathcal{M}_3$
$Q1$
$Q2$

Summary

- Fermat's principle says the path of a beam of light solves a *minimal time problem*.
- Bernoulli's approach to solving the Brachistochrone relied on solving a system with **piecewise constant dynamics** that approximates the actual dynamics.
- Elvis is solving a *minimal time problem* with **discontinuous dynamics**, and intuits Snell's Law.

These models lie outside existing theory, since

Almost all optimal control theory requires (at least) Lipschitz dynamics in the state variable.

Summary

- Fermat's principle says the path of a beam of light solves a *minimal time problem*.
- Bernoulli's approach to solving the Brachistochrone relied on solving a system with **piecewise constant dynamics** that approximates the actual dynamics.
- Elvis is solving a *minimal time problem* with **discontinuous dynamics**, and intuits Snell's Law.

These models lie outside existing theory, since

Almost all optimal control theory requires (at least) Lipschitz dynamics in the state variable.

Structural discontinuous systems

Stratified domains: The state space is partitioned into a finite collection $\{\mathcal{M}_1, \ldots, \mathcal{M}_M\}$ of smooth manifolds embedded in $\mathbb{R}^N$ such that

1. $\mathbb{R}^N = \bigcup_{i=1}^M \mathcal{M}_i$; $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$ for all $i \neq j$.
2. If $\overline{\mathcal{M}}_i \cap \mathcal{M}_j \neq \emptyset$, then $\mathcal{M}_j \subseteq \overline{\mathcal{M}}_i$.
3. Each $\overline{\mathcal{M}}_i$ is **proximally smooth** of radius $\delta > 0$;
4. Each $\overline{\mathcal{M}}_i$ is **relatively wedged**.

$\overline{\mathcal{M}}$ **Proximally smooth:** The distance function $d_{\overline{\mathcal{M}}}(x) := \inf_{y \in \overline{\mathcal{M}}} ||x - y||$ is differentiable on $\{\mathcal{M} + \delta\mathbb{B}\} \backslash \overline{\mathcal{M}}$. **One consequence:** The Clarke normal cone $\mathcal{N}_{\overline{\mathcal{M}}}(x)$ is the proximal one, and has closed graph.

$\overline{\mathcal{M}}$ **relatively wedged:** The dimension of the relative interior of the tangent cone $\mathcal{T}_{\overline{\mathcal{M}}}(x)$ is the dimension of $\mathcal{M}$ for all $x \in \overline{\mathcal{M}}$.

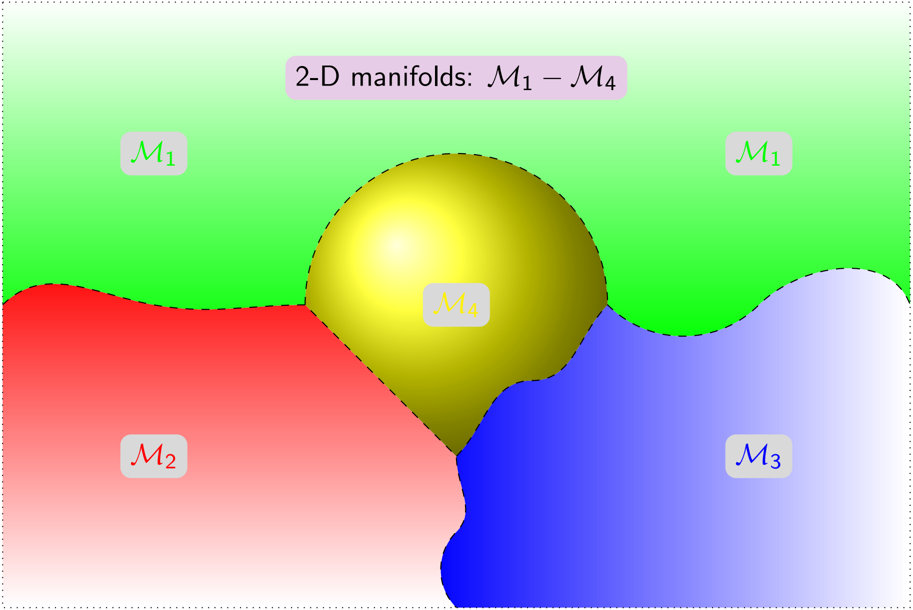
Structural discontinuous systems

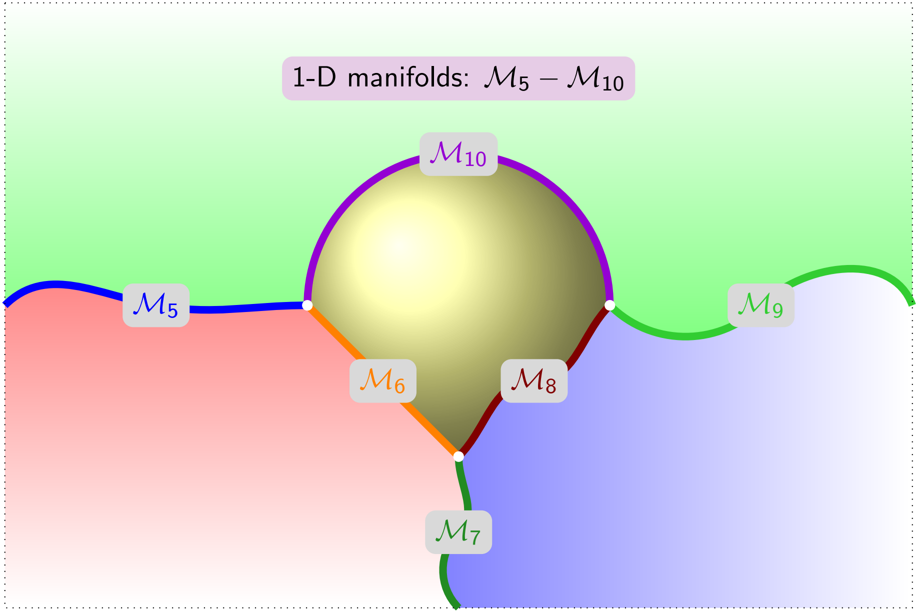
Stratified domains: The state space is partitioned into a finite collection $\{\mathcal{M}_1, \ldots, \mathcal{M}_M\}$ of smooth manifolds embedded in $\mathbb{R}^N$ such that

1. $\mathbb{R}^N = \bigcup_{i=1}^M \mathcal{M}_i$; $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$ for all $i \neq j$.
2. If $\overline{\mathcal{M}}_i \cap \mathcal{M}_j \neq \emptyset$, then $\mathcal{M}_j \subseteq \overline{\mathcal{M}}_i$.
3. Each $\overline{\mathcal{M}}_i$ is **proximally smooth** of radius $\delta > 0$;
4. Each $\overline{\mathcal{M}}_i$ is **relatively wedged**.

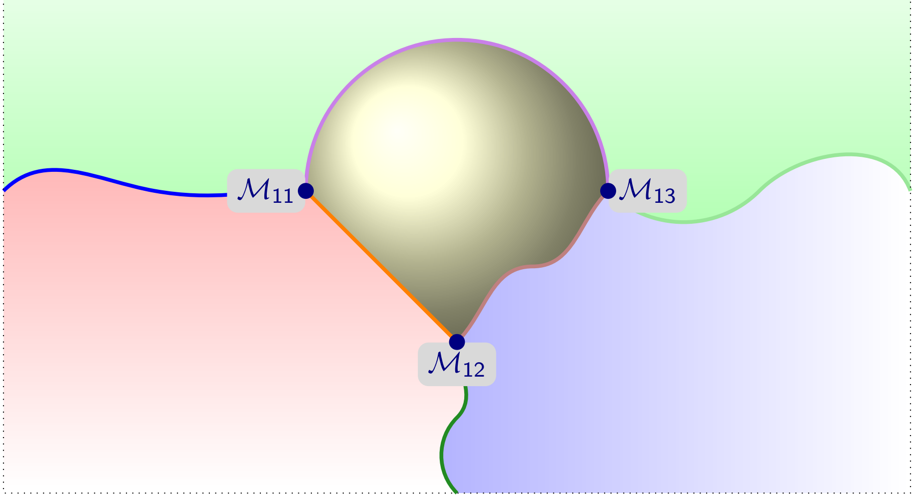
$\overline{\mathcal{M}}$ **Proximally smooth:** The distance function $d_{\overline{\mathcal{M}}}(x) := \inf_{y \in \overline{\mathcal{M}}} ||x - y||$ is differentiable on $\left\{\mathcal{M} + \delta\mathbb{B}\right\} \backslash \overline{\mathcal{M}}$. **One consequence:** The Clarke normal cone $\mathcal{N}_{\overline{\mathcal{M}}}(x)$ is the proximal one, and has closed graph.

$\overline{\mathcal{M}}$ **relatively wedged:** The dimension of the relative interior of the tangent cone $\mathcal{T}_{\overline{\mathcal{M}}}(x)$ is the dimension of $\mathcal{M}$ for all $x \in \overline{\mathcal{M}}$.


2-D manifolds: $\mathcal{M}_1 - \mathcal{M}_4$
$\mathcal{M}_1$
$\mathcal{M}_1$
$\mathcal{M}_4$
$\mathcal{M}_2$
$\mathcal{M}_3$


1-D manifolds: $\mathcal{M}_5 - \mathcal{M}_{10}$
$\mathcal{M}_{10}$
$\mathcal{M}_5$
$\mathcal{M}_9$
$\mathcal{M}_6$
$\mathcal{M}_8$
$\mathcal{M}_7$

0-D manifolds: $\mathcal{M}_{11} - \mathcal{M}_{13}$



The dynamics

Associated to each manifold $\mathcal{M}_i$ is a multifunction $F_i : \mathcal{M}_i \rightrightarrows \mathbb{R}^N$ for which $(\mathcal{M}_i, F_i)$ satisfies the Basic Assumptions **(BA)**:

$$\textbf{(BA)} \quad \begin{cases} \textbf{1)} \text{ gr } F(\cdot) := \big\{(x,v) : v \in F(x)\big\} \text{ is closed w.r.t. } \mathcal{M}, \\ \textbf{2)} \, \forall\, x \in \mathcal{M},\ F(x) \subseteq \mathcal{T}_{\mathcal{M}}(x) \text{ is nonempty, convex, and compact,} \\ \textbf{3)} \, \exists\ r > 0 \text{ so that } \max\{|v| : v \in F(x)\} \leq r(1+|x|), \textit{ and} \\ \textbf{4)} \, F(\cdot) \text{ is Lipschitz on bounded sets of } \mathcal{M}. \end{cases}$$

The **basic velocity** multifunction $F : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ is defined by

$$F(x) = F_i(x) \quad \text{whenever} \quad x \in \mathcal{M}_i.$$

This multifunction induces a differential inclusion with no general existence theory or compactness of trajectories. The **Krasovskii regularization** $G : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$

$$G(x) = \bigcap_{\varepsilon > 0} \overline{\text{co}} \bigcup \{F(y) : \|y - x\| < \varepsilon\} = \text{co} \bigcup_{x \in \overline{\mathcal{M}_i}} \overline{F}_i(x).$$

rectifies that issue, but is generally discontinuous on $\mathbb{R}^N$.

The dynamics

Associated to each manifold $\mathcal{M}_i$ is a multifunction $F_i : \mathcal{M}_i \rightrightarrows \mathbb{R}^N$ for which $(\mathcal{M}_i, F_i)$ satisfies the Basic Assumptions **(BA)**:

$$\textbf{(BA)} \quad \begin{cases} \textbf{1)}\ \text{gr } F(\cdot) := \{(x,v) : v \in F(x)\} \text{ is closed w.r.t. } \mathcal{M}, \\ \textbf{2)}\ \forall\, x \in \mathcal{M},\ F(x) \subseteq \mathcal{T}_{\mathcal{M}}(x) \text{ is nonempty, convex, and compact,} \\ \textbf{3)}\ \exists\, r > 0 \text{ so that } \max\{|v| : v \in F(x)\} \leq r(1+|x|), \textit{ and} \\ \textbf{4)}\ F(\cdot) \text{ is Lipschitz on bounded sets of } \mathcal{M}. \end{cases}$$

The **basic velocity** multifunction $F : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ is defined by

$$F(x) = F_i(x) \quad \text{whenever} \quad x \in \mathcal{M}_i.$$

This multifunction induces a differential inclusion with no general existence theory or compactness of trajectories. The **Krasovskii regularization** $G : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$

$$G(x) = \bigcap_{\varepsilon > 0} \overline{\text{co}} \bigcup \{F(y) : \|y - x\| < \varepsilon\} = \text{co} \bigcup_{x \in \overline{\mathcal{M}_i}} \overline{F}_i(x).$$

rectifies that issue, but is generally discontinuous on $\mathbb{R}^N$.

Proposed research program:

Extend the elements of optimal control to systems with stratified dynamics

A Structural Condition:

$$G(x) \cap \mathcal{T}_{\overline{\mathcal{M}}_i}(x) \subseteq \overline{F}_i(x) \cap \mathcal{T}_{\overline{\mathcal{M}}_i}(x) \quad \text{when } x \in \overline{\mathcal{M}}_i.$$

This (essentially) says a velocity just off a highway is allowed on the highway:

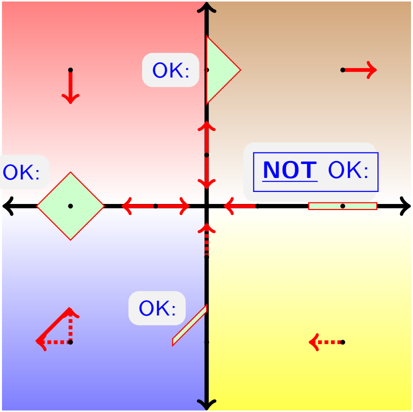


Figure : Illustration of Structural Condition

Minimal Time Problems

We are given a closed set $S \subseteq \mathbb{R}^N$ (the target), and a multifunction $G : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$. The dynamics are

$$\textbf{(DI)} \qquad \begin{cases} \dot{x}(t) \in G\big(x(t)\big) & \text{a.e. } t \in [0, T] \\ x(0) = x, \end{cases}$$

and the **minimal time problem** consists of

$$\min\ T \quad \text{over } x(\cdot) \text{ satisfying } \textbf{(DI)} \text{ with } x(T) \in S,$$

and $T(x)$ is the optimal value.

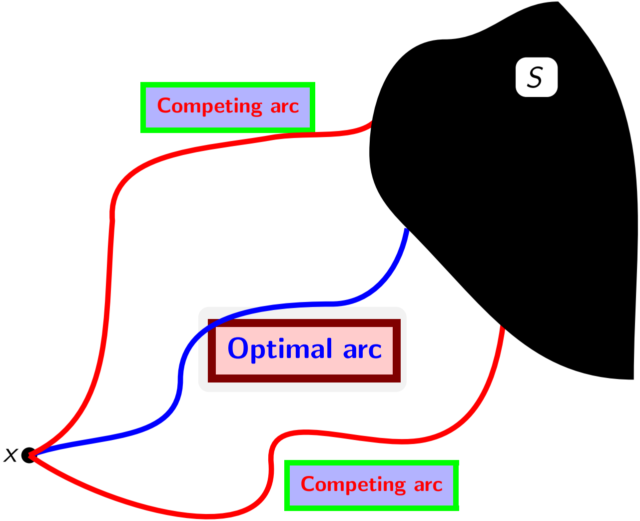


Figure : Minimal Time Problem

Reachable sets

$$\textbf{(DI)} \qquad \begin{cases} \dot{x}(t) \in G\big(x(t)\big) & \text{a.e. } t \in [0, T] \\ x(0) = x, \end{cases}$$

The fundamental object in Min Time problems is the **Reachable Set**

$$R_G^{(T)}(x) = \big\{ x(T) : \ x(\cdot) \text{ satisfies } \textbf{(DI)} \big\},$$

since

$$T(x) = \min\big\{ T : x \in R_{-G}^{(T)}(S) \big\}.$$

Characterizations of the multifunction $(t,x) \mapsto R_G^{(T)}(x)$

I. **Exponential Formula** **(PW '90)**

$$R_G^{(T)}(x) = \lim_{k\to\infty} \left(I + \frac{T}{k} G \right)^k (x).$$

II. **Semigroup characterization** **(PW '90)**

$T \mapsto R_G^{(T)}(\cdot)$ is a one-parameter semigroup with infinitesimal generator $G(\cdot)$.

III. **Funnel Equation** **(Panasyuk & Panasyuk '88)**

$$R^{(T+h)}(x) \approx \bigcup \{ y + hG(y) : \ y \in R^{(T)}(x) \}.$$

IV. **(HJ) equation** **(Clarke '00)** Let $\mathcal{R} = \{(T,y) : T \geq 0,\ y \in R_G^{(T)}(x)\}$ be the graph of $R_G^{(\cdot)}(x)$. Then

$$\sigma + H_G(y,\zeta) = 0 \quad \forall (T,y) \in \mathcal{R}, (\sigma,\zeta) \in N_{\mathcal{R}}^P(T,y).$$

All characterizations rely crucially on the Lipschitz assumption **(BA.4)**, so none of them carry over to stratified systems.

We focus here on a penetration result and extending it to stratified systems:

Theorem (Clarke-PW, '95)

Suppose $\mathcal{C} \subseteq \mathcal{M} \subseteq \mathbb{R}^N$ *is closed and* $(\mathcal{M}, F)$ *satisfies* **(BA)**. *Assume* $\mathcal{C} \subset \mathcal{M}$ *is wedged at* $x \in \mathcal{C}$ *and* $v \in \operatorname{int} \mathcal{T}_{\mathcal{C}}(x)$. *Then* $\exists C^1$ *trajectory* $x(\cdot)$ *of (DI) with* $\dot{x}(0) = v$ *and* $x(t) \in \mathcal{C}$ *for small* t.

What we need and can prove is more refined. The context is a stratified system.

Theorem (R.Barnard-PW, 2013)

Suppose $x \in \overline{\mathcal{M}}_i$ *and* $v \in \mathcal{T}_{\overline{\mathcal{M}}_i}(x) \cap \overline{F}_i(x)$. *Then there exists* C^1 *trajectory* $x(\cdot)$ *of (DI) with* $\dot{x}(0) = v$ *and* $x(t) \in \overline{\mathcal{M}}_i$ *for small* t.

All characterizations rely crucially on the Lipschitz assumption **(BA.4)**, so none of them carry over to stratified systems.

We focus here on a penetration result and extending it to stratified systems:

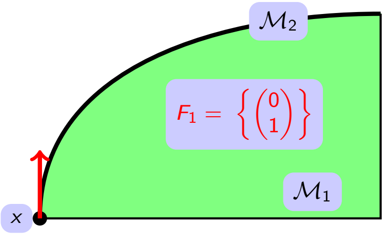
Theorem (Clarke-PW, '95)

Suppose $\mathcal{C} \subseteq \mathcal{M} \subseteq \mathbb{R}^N$ *is closed and* $(\mathcal{M}, F)$ *satisfies* **(BA)**. *Assume* $\mathcal{C} \subset \mathcal{M}$ *is wedged at* $x \in \mathcal{C}$ *and* $v \in int\, \mathcal{T}_{\mathcal{C}}(x)$. *Then* $\exists C^1$ *trajectory* $x(\cdot)$ *of (DI) with* $\dot{x}(0) = v$ *and* $x(t) \in \mathcal{C}$ *for small* t.

What we need and can prove is more refined. The context is a stratified system.

Theorem (R.Barnard-PW, 2013)

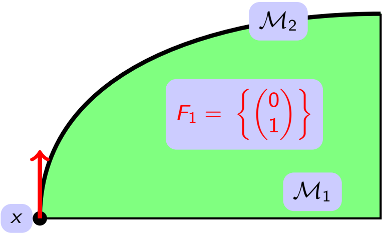
Suppose $x \in \overline{\mathcal{M}}_i$ *and* $v \in \mathcal{T}_{\overline{\mathcal{M}_i}}(x) \cap \overline{F}_i(x)$. *Then there exists* C^1 *trajectory* $x(\cdot)$ *of (DI) with* $\dot{x}(0) = v$ *and* $x(t) \in \overline{\mathcal{M}}_i$ *for small* t.



We have $v := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in F_1 \bigcap \mathcal{T}_{\overline{\mathcal{M}_1}}(x)$ and the trajectory cannot stay in $\overline{\mathcal{M}}_1$ using the velocity of F_1. However the structural condition implies $v \in \overline{F}_2$ and one observes $v \in r - \operatorname{int} \mathcal{T}_{\overline{\mathcal{M}_2}}(x)$, hence there exists a trajectory $x(\cdot)$ with $x(t) \in \overline{\mathcal{M}}_2$ for small t.

Structural condition:

$$G(x) \cap \mathcal{T}_{\overline{\mathcal{M}_i}}(x) \subseteq \overline{F}_i(x) \cap \mathcal{T}_{\overline{\mathcal{M}_i}}(x) \quad \text{when } x \in \overline{\mathcal{M}}_i.$$



We have $v := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in F_1 \bigcap \mathcal{T}_{\overline{\mathcal{M}_1}}(x)$ and the trajectory cannot stay in $\overline{\mathcal{M}}_1$ using the velocity of F_1. However the structural condition implies $v \in \overline{F}_2$ and one observes $v \in r - \operatorname{int} \mathcal{T}_{\overline{\mathcal{M}_2}}(x)$, hence there exists a trajectory $x(\cdot)$ with $x(t) \in \overline{\mathcal{M}}_2$ for small t.

Structural condition:

$$G(x) \cap \mathcal{T}_{\overline{\mathcal{M}_i}}(x) \subseteq \overline{F}_i(x) \cap \mathcal{T}_{\overline{\mathcal{M}_i}}(x) \quad \text{when } x \in \overline{\mathcal{M}}_i.$$

Essential velocities

Suppose $(\mathcal{M}, F)$ satisfies **(BA)**. Then for all $v \in F(x)$, there exists a trajectory $x(\cdot)$ of (DI) with $\dot{x}(0) = v$; i.e. every velocity at every point *matters* or is *essential* to the flow. This is clearly not the case for stratified systems.

Which velocities of a stratified system are essential?

Given $x \in \mathbb{R}^N$ and $v \in G(x)$, when does there exist a trajectory $x(\cdot)$ of $(\mathrm{DI})_G$ for which $\dot{x}(0) = v$?

Essential velocities

Suppose $(\mathcal{M}, F)$ satisfies **(BA)**. Then for all $v \in F(x)$, there exists a trajectory $x(\cdot)$ of (DI) with $\dot{x}(0) = v$; i.e. every velocity at every point *matters* or is *essential* to the flow. This is clearly not the case for stratified systems.

Which velocities of a stratified system are essential?

Given $x \in \mathbb{R}^N$ and $v \in G(x)$, when does there exist a trajectory $x(\cdot)$ of $(\mathrm{DI})_G$ for which $\dot{x}(0) = v$?

A Structural Condition:

$$G(x) \cap \mathcal{T}_{\overline{\mathcal{M}}_i}(x) \subseteq \overline{F}_i(x) \cap \mathcal{T}_{\overline{\mathcal{M}}_i}(x) \quad \text{when } x \in \overline{\mathcal{M}}_i.$$

This (essentially) says a velocity just off a highway is allowed on the highway:

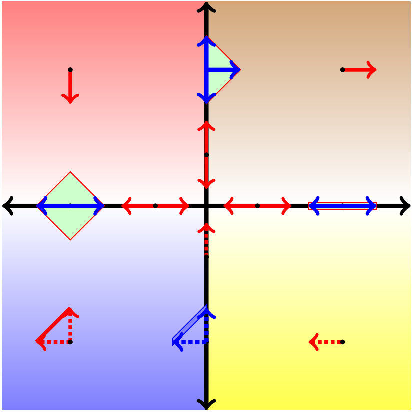


Figure : The essential velocities

We describe these sets where $\dim \mathcal{M}_i = N$, but the reasoning is valid for any dimension.

A closed set $\mathcal{C} \subseteq \mathbb{R}^N$ is **wedged** (aka epi-Lipschitz, directionally Lipschitz) at $x \in \mathcal{C}$ provided the following (equivalent) conditions hold:

- $N_{\mathcal{C}}(x)$ is pointed;
- $\operatorname{int} \mathcal{T}_{\mathcal{C}}(x) \neq \emptyset$;
- There exist a unitary map $\Phi : \mathbb{R}^N \to \mathbb{R}^{N-1} \times \mathbb{R}$, a constant $\varepsilon > 0$, and a Lipschitz function $g : \varepsilon\mathbb{B}^{N-1} \to \mathbb{R}$ so that $g(0) = 0$ and

$$\Big[\Phi(\mathcal{C} - x)\Big] \bigcap \left[\varepsilon\mathbb{B}^{N-1} \times (-\varepsilon, \varepsilon)\right] = [\text{epi } g] \bigcap \left[\varepsilon\mathbb{B}^{N-1} \times (-\varepsilon, \varepsilon)\right]$$

If we also assume $\mathcal{C}$ is proximally smooth, then the Lipschitz function $g(\cdot)$ is semiconvex, and so has a representation

$$g(x) = \max_{\alpha \in A} g_\alpha(x)$$

where each $g_\alpha : \varepsilon\mathbb{B}^{N-1} \to \mathbb{R}$ is C^2, and A is a compact metric space. (In our application where $\mathcal{C} = \overline{\mathcal{M}}$ is a stratified domain, A is a finite set.)

We describe these sets where $\dim \mathcal{M}_i = N$, but the reasoning is valid for any dimension.

A closed set $\mathcal{C} \subseteq \mathbb{R}^N$ is **wedged** (aka epi-Lipschitz, directionally Lipschitz) at $x \in \mathcal{C}$ provided the following (equivalent) conditions hold:

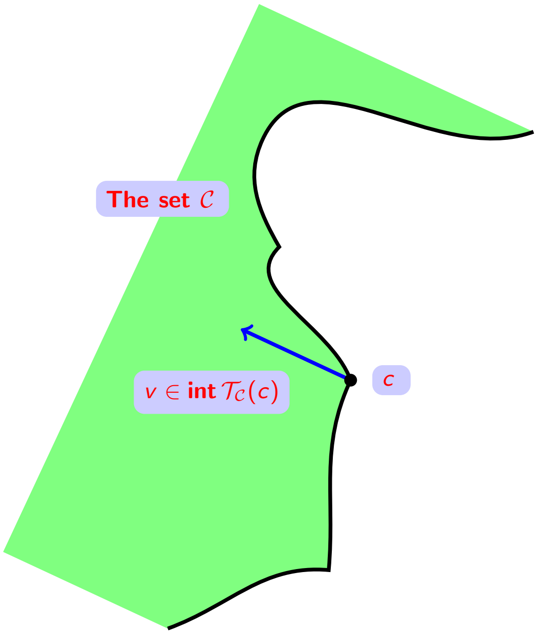
- $N_{\mathcal{C}}(x)$ is pointed;
- $\operatorname{int} \mathcal{T}_{\mathcal{C}}(x) \neq \emptyset$;
- There exist a unitary map $\Phi : \mathbb{R}^N \to \mathbb{R}^{N-1} \times \mathbb{R}$, a constant $\varepsilon > 0$, and a Lipschitz function $g : \varepsilon\mathbb{B}^{N-1} \to \mathbb{R}$ so that $g(0) = 0$ and

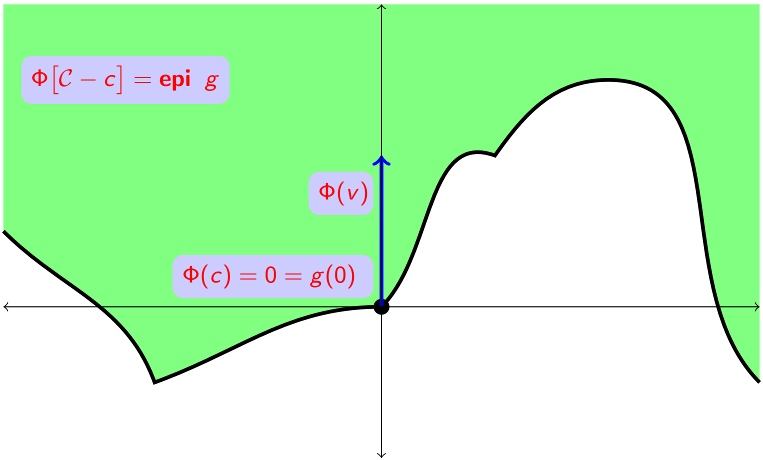
$$\Big[\Phi(\mathcal{C} - x)\Big] \bigcap \left[\varepsilon\mathbb{B}^{N-1} \times (-\varepsilon, \varepsilon)\right] = [\text{epi } g] \bigcap \left[\varepsilon\mathbb{B}^{N-1} \times (-\varepsilon, \varepsilon)\right]$$

If we also assume $\mathcal{C}$ is proximally smooth, then the Lipschitz function $g(\cdot)$ is semiconvex, and so has a representation

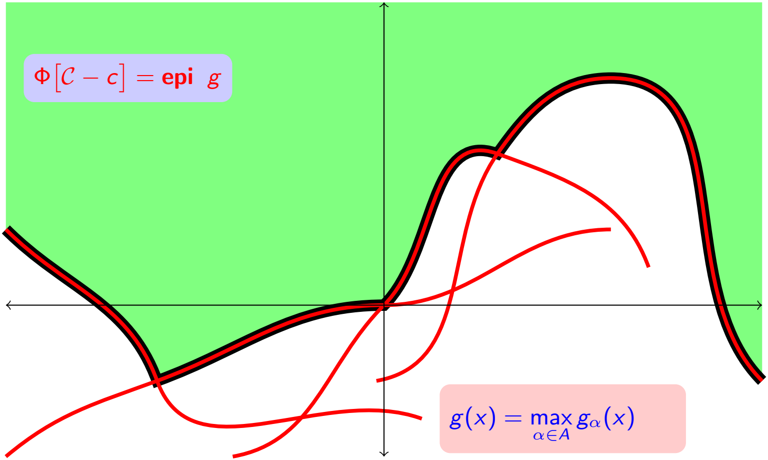
$$g(x) = \max_{\alpha \in A} g_\alpha(x)$$

where each $g_\alpha : \varepsilon\mathbb{B}^{N-1} \to \mathbb{R}$ is C^2, and A is a compact metric space. (In our application where $\mathcal{C} = \overline{\mathcal{M}}$ is a stratified domain, A is a finite set.)


The set $\mathcal{C}$
$v \in \textbf{int}\, \mathcal{T}_{\mathcal{C}}(c)$
c



The rotation by Φ transforms v to the vertical direction.



The relative boundary of $\mathcal{T}_{\overline{\mathcal{M}}}(x)$ is associated with the gradients of the g_α's, and belongs to the relative interior of the tangent cone to the manifold with boundary of lowest dimension of such a g_α.

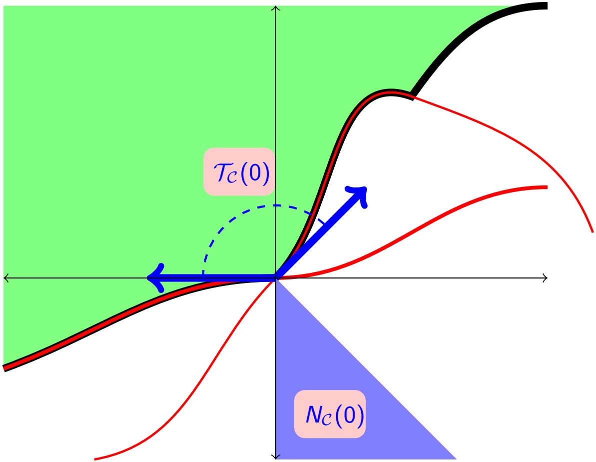
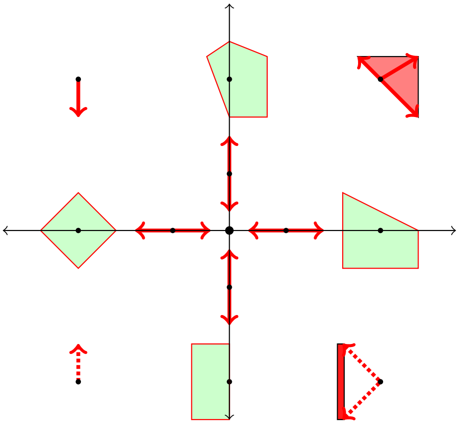


Figure : The relative boundary of $\mathcal{T}_{\mathcal{C}}(x)$

Definition

The essential multifunction $G^\sharp : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ *is given by*

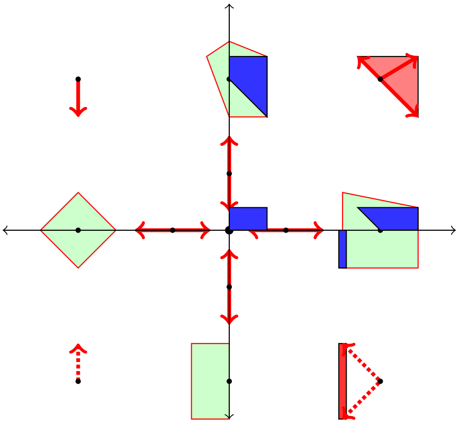
$$G^\sharp(x) := \bigcup_{i,\, x \in \overline{\mathcal{M}}_i} \left[\overline{F}_i(x) \cap \mathcal{T}_{\overline{\mathcal{M}}_i}(x) \right].$$



Definition

The essential multifunction $G^\sharp : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ *is given by*

$$G^\sharp(x) := \bigcup_{i,\, x \in \overline{\mathcal{M}}_i} \left[\overline{F}_i(x) \cap \mathcal{T}_{\overline{\mathcal{M}}_i}(x) \right].$$



Conclusions

- Bernoulli's Brachistochrone approximation scheme and modeling Elvis's fetching are examples of a minimal time problem with discontinuous dynamics. Stratified systems seems to be an attractive model for these and many real-life scenarios. The system is discontinuous, but it has structure that can be exploited.
- The basic philosophy is that the subsystems are very "nice" but the overall system does not fit any existing theory. The issue is to find how to patch the pieces together.
- This appears to be quite challenging, but is mostly wide open - even the case of constant dynamics where $F_i(x) = F_i$ (a fixed convex set). All the major issues - necessary conditions, HJ theory, feedback synthesis, regularity theory, numerics, etc. are not yet resolved.
- Finding essential velocities is a piece of the puzzle.

Conclusions

- Bernoulli's Brachistochrone approximation scheme and modeling Elvis's fetching are examples of a minimal time problem with discontinuous dynamics. Stratified systems seems to be an attractive model for these and many real-life scenarios. The system is discontinuous, but it has structure that can be exploited.
- The basic philosophy is that the subsystems are very "nice" but the overall system does not fit any existing theory. The issue is to find how to patch the pieces together.
- This appears to be quite challenging, but is mostly wide open - even the case of constant dynamics where $F_i(x) = F_i$ (a fixed convex set). All the major issues - necessary conditions, HJ theory, feedback synthesis, regularity theory, numerics, etc. are not yet resolved.
- Finding essential velocities is a piece of the puzzle.

Conclusions

- Bernoulli's Brachistochrone approximation scheme and modeling Elvis's fetching are examples of a minimal time problem with discontinuous dynamics. Stratified systems seems to be an attractive model for these and many real-life scenarios. The system is discontinuous, but it has structure that can be exploited.
- The basic philosophy is that the subsystems are very "nice" but the overall system does not fit any existing theory. The issue is to find how to patch the pieces together.
- This appears to be quite challenging, but is mostly wide open - even the case of constant dynamics where $F_i(x) = F_i$ (a fixed convex set). All the major issues - necessary conditions, HJ theory, feedback synthesis, regularity theory, numerics, etc. are not yet resolved.
- Finding essential velocities is a piece of the puzzle.

Thank you,

ever and

ever again,

Terry!