Necessary optimality conditions for optimal control problems with equilibrium constraints

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Thanks to my collaborator: Lei Guo

MP with Equilibrium Constraints

$$
\begin{array}{lll}
\text{(MPEC)} & \min & f(x) \\
& \text{s.t.} & \underbrace{G(x) \leq 0, H(x) \leq 0, G(x)^T H(x) = 0}_{\text{complementarity constraints}}
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MPEC is a very difficult nonconvex optimization problem since the usual constraint qualification such as the Mangasarian-Fromovitz CQ is violated at any feasible point of MPEC. The classical KKT condition may not hold.

Reformulations of MPECs

The complementarity (equilibrium) constraint

$$G(x) \leq 0, H(x) \leq 0, G(x)^\top H(x) = 0$$

can be reformulated as a geometric constraint

$$(G(x), H(x)) \in \Omega \text{ where } \Omega := \{(y, z) | 0 \geq y \perp z \leq 0\},$$

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can be reformulated as a geometric constraint

$$(G(x), H(x)) \in \Omega \text{ where } \Omega := \{(y, z) | 0 \geq y \perp z \leq 0\},$$

or as a nonsmooth equation constraint

$$\max\{G(x), H(x)\} = 0.$$

The regular normal cone to Ω at $(\bar{y}, \bar{z}) \in \Omega$:

$$\widehat{\mathcal{N}}_\Omega(\bar{y}, \bar{z}) = \left\{ (\xi, \gamma) : \begin{array}{lll} \xi_i = 0 & \textit{if} & \bar{y}_i < 0, \bar{z}_i = 0 \\ \gamma_i = 0 & \textit{if} & \bar{z}_i < 0, \bar{y}_i = 0 \\ \xi_i \geq 0, \gamma_i \geq 0 & \textit{if} & \bar{y}_i = \bar{z}_i = 0 \end{array} \right\}.$$

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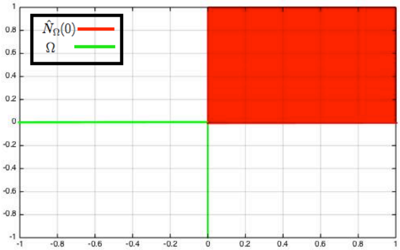
The limiting normal cone of Ω at $(\bar{y}, \bar{z}) \in \Omega$ is

$$\mathcal{N}_\Omega(\bar{y}, \bar{z}) = \left\{ (\xi, \gamma) : \begin{array}{ll} \xi_i = 0 & \textit{if} \quad \bar{y}_i < 0 \\ \gamma_i = 0 & \textit{if} \quad \bar{z}_i < 0 \\ \left\{ \begin{array}{ll} \textit{either} & \xi_i > 0, \gamma_i > 0 \\ \textit{or} & \xi_i \gamma_i = 0 \end{array} \right. & \textit{if} \quad \bar{y}_i = \bar{z}_i = 0 \end{array} \right\}.$$


$\hat{N}_\Omega(0)$
Ω
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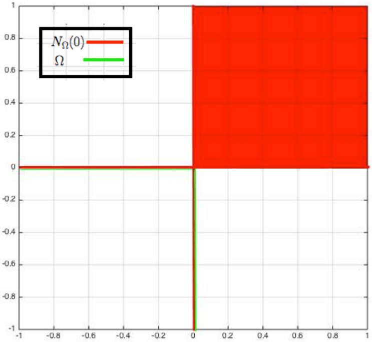
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Strong stationary condition

Reformulate MPEC as:

$$\text{(EMPEC)} \qquad \min \quad f(x) \qquad s.t. \quad (G(x), H(x)) \in \Omega.$$

Let $\bar{x}$ be a local solution of (MPEC). If MPEC LICQ holds at $\bar{x}$, that is, the gradients of all active constraints

$$\nabla G_i(\bar{x})(i \in I_G(\bar{x})), \nabla H_i(\bar{x})(i \in I_H(\bar{x}))$$

are linearly independent, then there exist multipliers (λ^G, λ^H) such that

$$0 = \nabla f(\bar{x}) + \nabla G(\bar{x})^T \lambda^G + \nabla H(\bar{x})^T \lambda^H, \quad (\lambda^G, \lambda^H) \in \widehat{\mathcal{N}}_\Omega(G(\bar{x}), H(\bar{x}))$$

S-stationary condition

Let $\Omega := \{(y, z) : 0 \geq y \perp z \leq 0\}$ be the complementarity cone. Since $(\lambda^G, \lambda^H) \in \widehat{\mathcal{N}}_\Omega(G(\bar{x}), H(\bar{x}))$ if and only if

$$\begin{array}{ll} \lambda_i^G = 0 & \textit{if} \quad H_i(\bar{x}) = 0, G_i(\bar{x}) < 0 \\ \lambda_i^H = 0 & \textit{if} \quad H_i(\bar{x}) < 0, G_i(\bar{x}) = 0 \\ \lambda_i^G \geq 0, \lambda_i^H \geq 0 & \textit{if} \quad G_i(\bar{x}) = H_i(\bar{x}) = 0 \end{array}.$$

The S-stationary condition is

$$0 = \nabla f(\bar{x}) + \nabla G(\bar{x})^\top \lambda^G + \nabla H(\bar{x})^\top \lambda^H$$
$$\lambda_i^G = 0 \text{ for } i \text{ such that } G_i(\bar{x}) < 0$$
$$\lambda_i^H = 0 \text{ for } i \text{ such that } H_i(\bar{x}) < 0$$
$$\lambda_i^G \geq 0, \lambda_i^H \geq 0 \text{ if } H_i(\bar{x}) = G_i(\bar{x}) = 0.$$

Mordukhovich stationary condition

Reformulate MPEC as:

$$\text{(EMPEC)} \qquad \min \quad f(x) \qquad s.t. \quad (G(x), H(x)) \in \Omega.$$

Let $\bar{x}$ be a local minimizer and suppose the problem is calm at $\bar{x}$. Then

$$0 = \nabla f(\bar{x}) + \nabla G(\bar{x})^T \lambda^G + \nabla H(\bar{x})^T \lambda^H, \quad (\lambda^G, \lambda^H) \in \mathcal{N}_\Omega(G(\bar{x}), H(\bar{x})).$$

M-stationary condition

Let $\Omega := \{(y,z) : 0 \geq y \perp z \leq 0\}$ be the complementarity cone. Since $(\lambda^G, \lambda^H) \in \mathcal{N}_\Omega(G(\bar{x}), H(\bar{x}))$ if and only if

$$\begin{array}{ll} \lambda_i^G = 0 & \textit{if} \quad H_i(\bar{x}) = 0, G_i(\bar{x}) < 0 \\ \lambda_i^H = 0 & \textit{if} \quad H_i(\bar{x}) < 0, G_i(\bar{x}) = 0 \\ \text{either } \lambda_i^G > 0, \lambda_i^H > 0 \text{ or } \lambda_i^G \lambda_i^H = 0 & \textit{if} \quad G_i(\bar{x}) = H_i(\bar{x}) = 0 \end{array}.$$

The M-stationary condition is

$$\begin{array}{l} 0 = \nabla f(\bar{x}) + \nabla G(\bar{x})^\top \lambda^G + \nabla H(\bar{x})^\top \lambda^H \\ \lambda_i^G = 0 \text{ for } i \text{ such that } G_i(\bar{x}) < 0 \\ \lambda_i^H = 0 \text{ for } i \text{ such that } H_i(\bar{x}) < 0 \\ \text{either } \lambda_i^G > 0, \lambda_i^H > 0 \text{ or } \lambda_i^G \lambda_i^H = 0 \text{ if } H_i(\bar{x}) = G_i(\bar{x}) = 0. \end{array}$$

Clarke stationary condition

$\min f(x)$ s.t.$\Phi_i(x) := \max\{G_i(x), H_i(x)\} = 0 \quad i = 1, \ldots, m$. By the nonsmooth KKT condition, if the problem is calm at $\bar{x}$, then $\exists$ λ such that $0 \in \nabla f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial\Phi_i(\bar{x})$. Since

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$$\partial \Phi_i(\bar{x}) = \begin{cases} \nabla G_i(\bar{x}) & \text{if } H_i(\bar{x}) < 0 \\ \nabla H_i(\bar{x}) & \text{if } G_i(\bar{x}) < 0 \\ \{\alpha \nabla G_i(\bar{x}) + (1-\alpha)\nabla H_i(\bar{x}) : \alpha \in [0,1]\} & \text{if } G_i = H_i = 0, \end{cases}$$

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we can find $\alpha_i \in [0, 1]$ such that

$$0 = \nabla f(\bar{x}) + \sum_{i=1}^m \underbrace{\alpha_i \lambda_i}_{\lambda_i^G} \nabla G_i(\bar{x}) + \sum_{i=1}^m \underbrace{(1-\alpha_i)\lambda_i}_{\lambda_i^H} \nabla H_i(\bar{x}).$$

The sign condition is $\lambda_i^G \lambda_i^H = \alpha_i(1-\alpha_i)\lambda_i^2 \geq 0$.

W-,S-,M- and C- stationary condition

A feasible $\bar{x}$ is called a weak stationary point if

$$0 = \nabla f(\bar{x}) + \nabla G(\bar{x})^\top \lambda^G + \nabla H(\bar{x})^\top \lambda^H$$
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$\bar{x}$ is a S-, M-, C- stationary point if it is a W-stationary and

$$\underbrace{\lambda_i^G \geq 0, \lambda_i^H \geq 0}_{\text{S Stationary}} \text{ if } H_i(\bar{x}) = G_i(\bar{x}) = 0$$

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In general we have S $\Longrightarrow$ M$\Longrightarrow$ C $\Longrightarrow$ W

Optimal control problems

Given a multifunction $U : [t_0, t_1] \rightrightarrows \mathbb{R}^m$, and a dynamic function $\phi : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. A control u is a measurable function on $[t_0, t_1]$ satisfying $u(t) \in U(t), a.e.$ and the state corresponding to a given control u, refers to a solution x of

$$\dot{x}(t) = \phi(t, x(t), u(t)) \quad \text{a.e.} \quad t \in [t_0, t_1],$$
$$(x(t_0), x(t_1)) \in E,$$

where E is a given subset in $\mathbb{R}^n \times \mathbb{R}^n$. Such a pair (x, u) is called an admissible pair. A standard optimal control problem is to find an admissible pair (x, u) such that an objective function $J(x, u)$ is minimized.

Optimal control problem with equilibrium constraints

(OCPEC):

$$
\begin{aligned}
\min \quad & J(x,u) := f(x(t_0), x(t_1)) \\
s.t. \quad & \dot{x}(t) = \phi(t, x(t), u(t)) \qquad a.e.\, t \in [t_0, t_1] \\
& 0 \le G(t, x(t), u(t)) \perp H(t, x(t), u(t)) \ge 0 \;\; a.e.\, t \in [t_0, t_1] \\
& (x(t_0), x(t_1)) \in E.
\end{aligned}
$$

Here for simplicity we have omitted the possible inequality and equality constraints and the control constraint $u(t) \in U(t)$.

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The equilibrium constraint includes the differential complementarity constraint introduced by Jong-Shi Pang and David Stewart (2008) as a special case.

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When $t_0 = t_1, \phi \equiv 0, x = u$, OCPEC becomes MPEC.

Bilevel Optimal Control Problems-Dynamic Stackelberg Games

The leader's decision variable is a vector z and the follower's problem is an optimal control problem.

$$
\begin{array}{llll}
P_2(z) & \min & g(x(t_1)) & \\
 & s.t. & \dot{x}(t) = \phi(t, x(t), u(t), z) & a.e.t \in [t_0, t_1] \\
 & & x(0) = x_0 & \\
 & & u(t) \in U(t) &
\end{array}
$$

The leader's problem:

$$P_1 : \quad \min f(x_z(t_1))$$

over all z and all optimal pairs (x_z, u_z) of $P_2(z)$.

The first order approach

If (x, u) is a solution of the follower's problem $P_2(z)$, then by Pontryagain maximum principle, there exists arc $p(t)$ such that

$$-\dot{p}(t) = \nabla_x H(t, x(t), u(t), z, p(t)), \ -p(t_1) = \nabla g(x(t_1))$$
$$\max_{u \in U(t)} H(t, x(t), u, z, p(t)) = H(t, x(t), u(t), z, p(t))$$

where the Hamiltonian $H(t, x, u, z, p) := \phi(t, x, u, z) \cdot p$.

The first order approach: replacing the maximum Hamiltonian condition by its necessary optimality condition. Suppose $U(t) := \{u : F(t, u) \leq 0\}$.

$$
\begin{array}{lll}
FP & \min & f(x(t_1)) \\
 & s.t. & \dot{x}(t) = \phi(t, x(t), u(t), z) \\
 & & -\dot{p}(t) = \nabla_x H(t, x(t), u(t), z, p(t)) \\
 & & \nabla_u H(t, x(t), u(t), z) + \nabla_u F(t, u(t))^T \lambda(t) = 0, \\
 & & 0 \geq F(t, u(t)) \perp -\lambda(t) \leq 0, \\
 & & x(0) = x_0, -p(t_1) = \nabla g(x(t_1)),
\end{array}
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 & & 0 \geq F(t, u(t)) \perp -\lambda(t) \leq 0, \\
 & & x(0) = x_0, -p(t_1) = \nabla g(x(t_1)),
\end{array}
$$

This is an optimal control problem with equilibrium constraints!

Dynamic Nash Equilibria

Each player solves an optimal control problem:

$$\begin{array}{lll} \min\limits_{x^\nu, u^\nu} & g_\nu(x^\nu, x^{-\nu}, u^\nu, u^{-\nu}) & \\ s.t. & \dot{x}^\nu(t) = \phi(x^\nu(t), u^\nu(t)) & a.e. t \in [t_0, t_1] \\ & x^\nu(0) = x_0^\nu, & \\ & u^\nu \in U^\nu. & \end{array}$$

If the control set U^ν is expressed as inequality constraints then the maximized Hamiltonian condition will lead to complementarity constraints in the necessary optimality condition.

Dynamic Stackelberg-Cournot-Nash equilibria

Stackelberg-Cournot-Nash equilibrium: N players (Followers) competing to optimize their own objectives to reach an Nash equilibria and the $N+1$ th player (Leader) optimizes its own objective taking into account the reactions of the other N firms and the effects of their reactions

$$
\begin{aligned}
\max_{x,u,\theta} \quad & f(x,u,\theta) \\
\text{s.t.} \quad & (x^\nu, u^\nu) \text{ maximizes } g_\nu(x,u,\theta) \\
& \qquad \text{s.t } \dot{x}^\nu = \phi(x^\nu(t), u^\nu(t)) \\
& \qquad\qquad x^\nu(0) = x_0^\nu, u^\nu \in U^\nu.
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$$\begin{aligned} \max_{x,u,\theta} \quad & f(x,u,\theta) \\ \text{s.t.} \quad & (x^\nu, u^\nu) \text{ maximizes } g_\nu(x,u,\theta) \\ & \qquad\qquad \text{s.t } \dot{x}^\nu = \phi(x^\nu(t), u^\nu(t)) \\ & \qquad\qquad\qquad x^\nu(0) = x_0^\nu, u^\nu \in U^\nu. \end{aligned}$$

If the control sets are inequality constraints, using the Pontryagain maximum principle to replace the followers' problem, we have an OCPEC!

In general for dynamic optimization problems, necessary optimality conditions need much stronger constraint qualifications and have weaker conclusions than static optimization problems!

Difficulites of generalizing from MPEC to OCPEC

If we formulate the complementarity constraint as $\Phi = \max\{G, H\} = 0$, then Φ is nonsmooth. No results in the current literature can be used to deal with nonsmooth equality constraints. If we treat $G \le 0, H \le 0, GH = 0$ as inequality and equality constraints, the current literature requires Mangasarian Fromovitz condition which never hold. If we reformulate the complementarity constraint as

$$\Psi := (G, H) \in \Omega := \{(y, z) : 0 \ge y \perp z \le 0\}$$

then the Euler inclusion includes the Clarke normal cone

$$(\dot{p}, 0) \in -\nabla\phi^T p + \nabla\Psi^T \eta, \quad \eta \in \mathcal{N}^c_\Omega.$$

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$$(\dot{p}, 0) \in -\nabla\phi^T p + \nabla\Psi^T \eta, \quad \eta \in \mathcal{N}_\Omega^c.$$

But $\mathcal{N}_\Omega^c$ is the whole space when $G_i = H_i = 0$.

Optimal control problems with mixed constraints

In their recent paper "Optimal control problems with mixed constraints", SICON 2010, Clarke and de Pinho consider the optimal control problems with mixed constraints:

$$
\begin{array}{lll}
(P) & \min\limits_{x,u} & J(x,u) := f(x(t_0), x(t_1)) \\
 & \text{s.t.} & \dot{x}(t) = \phi(t, x(t), u(t)) \quad \text{a.e.} \quad t \in [t_0, t_1], \\
 & & (x(t), u(t)) \in S(t) \quad \text{a.e.} \quad t \in [t_0, t_1], \\
 & & (x(t_0), x(t_1)) \in E,
\end{array}
$$

where $S(t)$ be a $\mathcal{L}$ measurable set-valued mapping.

Concept of a local minimum

Given a measurable radius function $R : [t_0, t_1] \to (0, \infty]$. A local minimizer of radius R for problem (P) is an admissible pair (x_*, u_*) that minimizes $J(x, u)$ over all admissible pairs (x, u) which satisfies

$$\int_{t_0}^{t_1} \|\dot{x}(t) - \dot{x}_*(t)\| dt \leq \varepsilon, \|x - x_*\|_\infty \leq \varepsilon, \|u(t) - u_*(t)\| \leq R(t).$$

In the case where $R(t) = \infty$, the concept coincides with the $W^{1,1}$ local minimum and when $R(t) = \varepsilon = \infty$, it becomes the global minimum.

Clarke and de Pinho's Theorem

Let (x^*, u^*) be a local minimizer of radius R for problem (P).

$$S_*^{\epsilon,R}(t) := \{(x,u) \in S(t) \mid \|x - x^*(t)\| \le \epsilon, \|u - u^*(t)\| \le R(t)\}.$$

Suppose that there exists a measurable function k_S such that, for almost every $t \in [t_0, t_1]$, the bounded slope condition holds, that is,

$$(x,u) \in S_*^{\epsilon,R}(t),\ (\alpha,\beta) \in \mathcal{N}^P_{S(t)}(x,u) \Longrightarrow \|\alpha\| \le k_S(t)\|\beta\|.$$

Assume further that there exist measurable functions k_x^ϕ, k_u^ϕ such that for almost every t and every $(x_i, u_i) \in S_*^{\epsilon,R}(t)$,

$$\|\phi(t,x_1,u_1) - \phi(t,x_2,u_2)\| \le k_x^\phi(t)\|x_1 - x_2\| + k_u^\phi(t)\|u_1 - u_2\|$$

the functions $\{k_x^\phi, k_S k_u^\phi\}$ are integrable and there exists $\eta > 0$ such that $R(t) \ge \eta k_S(t)$ for almost every $t \in [t_0, t_1]$.

Then there exist a number $\lambda_0 \in \{0, 1\}$ and an arc p such that
(i) the nontriviality condition holds: $(\lambda_0, p(t)) \neq 0 \quad \forall t \in [t_0, t_1]$;
(ii) the transversality condition holds:
$(p(t_0), -p(t_1)) \in \lambda_0 \nabla f(x^*(t_0), x^*(t_1)) + \mathcal{N}_E(x^*(t_0), x^*(t_1))$;
(iii) the Euler adjoint inclusion holds: for almost every $t \in [t_0, t_1]$,

$$(\dot{p}(t), 0) \quad \in \quad -\nabla \phi(t, x^*(t), u^*(t))^T p(t) + \mathcal{N}^c_{S(t)}(x^*(t), u^*(t));$$

(iv) the Weierstrass condition for radius R: for almost every $t \in [t_0, t_1]$,

$$(x_*(t), u) \in S(t), \;\; \|u - u^*(t))\| \leq R(t)$$
$$\Longrightarrow \langle p(t), \phi(t, x^*(t), u) \rangle \leq \langle p(t), \phi(t, x^*(t), u^*(t)) \rangle.$$

An improved Clarke and De Pinho's Theorem

We can show that in the Clarke and De Pinho's theorem, the Euler adjoint inclusion can be improved to:
(iii) for almost every $t \in [t_0, t_1]$,

$$(\dot{p}(t), 0) \in -\nabla\phi(t, x^*(t), u^*(t))^T p(t) + \mathrm{co}\, \mathcal{N}_{S(t)}(x^*(t), u^*(t));$$

The above Euler adjoint inclusion is sharper than the one in the Clarke and De Pinho's theorem in that the Clarke normal cone is replaced by the convex hull of the limiting normal cone since the inclusion
$\mathrm{co}\, \mathcal{N}_{S(t)}(x^*(t), u^*(t)) \subset \mathcal{N}^C_{S(t)}(x^*(t), u^*(t))$ **may be strict!**

We reformulate the problem OCPEC as

$$
\begin{array}{llll}
(P) & \min\limits_{x,u} & J(x,u) & \\
& \text{s.t.} & \dot{x}(t) = \phi(t, x(t), u(t)) \quad \text{a.e.} \quad t \in [t_0, t_1], \\
& & (x(t), u(t)) \in S(t) \quad \text{a.e.} \quad t \in [t_0, t_1], \\
& & (x(t_0), x(t_1)) \in E,
\end{array}
$$

where

$$S(t) := \left\{(x, u) \mid (G(t, x, u), H(t, x, u)) \in \Omega\right\}$$

with $\Omega := \{(y, z) | 0 \geq y \perp z \leq 0\}$.

Stationary conditions for OCPEC

Theorem: Assume that (x^*, u^*) is a local minimizer of radius R for problem (OCPEC). Suppose there exists a measurable function k_S such that, for almost every $t \in [t_0, t_1]$, the local error bound condition holds for the system

$$(G(t, x, u), H(t, x, u)) \in \Omega$$

at $(x^*(t), u^*(t))$ and the bounded slope condition holds. Assume further that the functions $\{k_x^\phi, k_S k_u^\phi\}$ are integrable and there exists $\eta > 0$ such that $R(t) \geq \eta k_S(t)$ for almost every $t \in [t_0, t_1]$.

Then there exist a number $\lambda_0 \in \{0, 1\}$, an arc p such that
(i) the nontriviality condition holds: $(\lambda_0, p(t)) \neq 0 \quad \forall t \in [t_0, t_1]$;
(ii) the transversality condition holds:
$(p(t_0), -p(t_1)) \in \lambda_0 \nabla f(x^*(t_0), x^*(t_1)) + \mathcal{N}_E(x^*(t_0), x^*(t_1))$.
(iv) the Weierstrass condition for radius R hold:

$$(x_*(t), u) \in S(t), \ \|u - u^*(t))\| \leq R(t)$$
$$\Longrightarrow \langle p(t), \phi(t, x^*(t), u)\rangle \leq \langle p(t), \phi(t, x^*(t), u^*(t))\rangle.$$

W-stationary condition for OCPEC

Moreover there exist the first multipliers $\eta^G(t), \eta^H(t)$ which are measurable functions such that (iii) the Euler adjoint inclusion holds: for almost every $t \in [t_0, t_1]$

$$
\begin{aligned}
-\dot{p}(t) = {} & \nabla_x \phi(t, x^*(t), u^*(t))^T p(t) + \nabla_x G(t, x^*(t), u^*(t))^T \eta^G(t) \\
& + \nabla_x H(t, x^*(t), u^*(t))^T \eta^H(t) \\
0 = {} & \nabla_u \phi(t, x^*(t), u^*(t))^T p(t) + \nabla_u G(t, x^*(t), u^*(t))^T \eta^G(t) \\
& + \nabla_u H(t, x^*(t), u^*(t))^T \eta^H(t) \\
\eta_i^G(t) = {} & 0 \text{ if } G_i > 0, \eta_i^H(t) = 0 \text{ if } H_i > 0.
\end{aligned}
$$

C- stationary condition

If for almost every $t \in [t_0, t_1]$, the problem

$$\begin{array}{ll} \min\limits_{u} & -\langle p(t), \phi(t, x^*(t), u)\rangle \\ \text{s.t.} & \max\{G(t, x^*(t), u), H(t, x^*(t), u)\} = 0. \end{array}$$

is calm at $u^*(t)$, then there exist the second multipliers λ^G, λ^H such that

$$\begin{aligned} 0 = {} & \nabla_u \phi(t, x^*(t), u^*(t))^T p(t) + \nabla_u G(t, x^*(t), u^*(t))^T \lambda^G(t) \\ & + \nabla_u H(t, x^*(t), u^*(t))^T \lambda^H(t) \\ & \lambda_i^G(t) = 0 \text{ if } G_i > 0, \lambda_i^H(t) = 0 \text{ if } H_i > 0 \\ & \lambda_i^G(t)\lambda_i^H(t) \geq 0 \text{ if } G_i = H_i = 0 \end{aligned}$$

M- stationary condition for OCPEC

If for almost every $t \in [t_0, t_1]$ the problem

$$\begin{array}{ll} \min_{u} & -\langle p(t), \phi(t, x^*(t), u)\rangle \\ \text{s.t.} & 0 \geq G(t, x^*(t), u) \perp H(t, x^*(t), u) \leq 0. \end{array}$$

is calm at $u^*(t)$, then there exist the second multipliers λ^G, λ^H such that

$$\begin{aligned} 0 = {} & \nabla_u \phi(t, x^*(t), u^*(t))^T p(t) + \nabla_u G(t, x^*(t), u^*(t))^T \lambda^G(t) \\ & + \nabla_u H(t, x^*(t), u^*(t))^T \lambda^H(t) \\ & \lambda_i^G(t) = 0 \text{ if } G_i > 0, \lambda_i^H(t) = 0 \text{ if } H_i > 0 \\ & \text{either } \lambda_i^G(t) > 0, \lambda_i^H(t) > 0 \text{ or } \lambda_i^G \lambda_i^H = 0 \text{ if } G_i = H_i = 0 \end{aligned}$$

S- stationary condition for OCPEC

If for almost every $t \in [t_0, t_1]$ the problem

$$\begin{aligned} \min_u \quad & -\langle p(t), \phi(t, x^*(t), u)\rangle \\ \text{s.t.} \quad & G(t, x^*(t), u) \leq 0, \quad H(t, x^*(t), u) \leq 0, \\ & G(t, x^*(t), u)^T H(t, x^*(t), u) \leq 0 \end{aligned}$$

is calm at $u^*(t)$, then there exist the second multipliers λ^G, λ^H such that

$$\begin{aligned} 0 = \nabla_u \phi(t, x^*(t), u^*(t))^T p(t) + \nabla_u G(t, x^*(t), u^*(t))^T \lambda^G(t) \\ + \nabla_u H(t, x^*(t), u^*(t))^T \lambda^H(t) \end{aligned}$$

$$\lambda_i^G(t) = 0 \text{ if } G_i > 0, \lambda_i^H(t) = 0 \text{ if } H_i > 0$$

$$\lambda_i^G(t) \geq 0, \lambda_i^H(t) \geq 0 \text{ if } G_i = H_i = 0$$

S-stationary condition for OCPEC under MPEC LICQ

In addition if for almost every $t \in [t_0, t_1]$, the MPEC LICQ holds at $u^*(t)$, i.e., the family of the gradients of the active constraints

$$\nabla_u G_i(t, x^*(t), u^*(t)) \ (i \in I_G) \qquad \nabla_u H_i(t, x^*(t), u^*(t)) \ (i \in I_H)$$

are linearly independent, then the first and the second multipliers coincide and the first multipliers also satisfy the sign condition:

$$\eta_i^G(t) \geq 0, \eta_i^H(t) \geq 0 \text{ if } G_i = H_i = 0.$$

References

Lei Guo and J.J. Ye, Necessary optimality conditions for optimal control problems with equilibrium constraints.

Happy birthday, Terry!