A class of total variation minimization problems

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The problem

- Given two increasing functions F and G on $[0,\infty)$ such that $F(0)=G(0)=0$ and $\lim_{t\to\infty} G(t)=\infty$, what are the infimum value $m_\pm$ and the minimizers u of the variational problem

$$(\mathcal{P}) \quad m_\pm := \inf\left\{\int_{\mathbb{R}^n} \mathrm{d}|\nabla u| \pm \int_{\mathbb{R}^n} F(|u|) : \int_{\mathbb{R}^n} G(|u|) = 1\right\}$$

- $n \geq 2$ is a given integer, and

$$\int_{\mathbb{R}^n} \mathrm{d}|\nabla u| = \|\nabla u\|_{\mathcal{M}(\mathbb{R}^n)}$$

denotes the total variation of a function $u : \mathbb{R}^n \to \mathbb{R}$, that is,

$$\|\nabla u\|_{\mathcal{M}(\mathbb{R}^n)} := \sup\left\{\int_{\mathbb{R}^n} u\, \mathrm{div}\phi : \ \phi \in C^1_c(\mathbb{R}^n;\mathbb{R}),\ |\phi(x)| \leq 1\right\}$$

- The mininization in problem $(\mathcal{P})$ is performed over functions $u : \mathbb{R}^n \to \mathbb{R}$ having finite total variation $\|\nabla u\|_{\mathcal{M}(\mathbb{R}^n)} < \infty$, and satisfying the constraint $\int_{\mathbb{R}^n} G(|u|) = 1$.

Motivation

Problem $(\mathcal{P})$ is motivated by the sharp L^1 Gagliardo-Nirenberg inequalities, which can be obtained from the L^1-Sobolev inequality and an interpolation inequality.

- **Sharp L^1 Sobolev inequality.** Denote $1^* = \frac{n}{n-1}$, then

$$\|u\|_{L^{1^*}(\mathbb{R}^n)} \leq (n\gamma_n^{1/n})^{-1}\|\nabla u\|_{\mathcal{M}(\mathbb{R}^n)} \quad \forall u \in \mathrm{BV}(\mathbb{R}^n) \qquad (1)$$

  where γ_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$.
- The best constant in the L^1 Sobolev inequality is $(n\gamma_n^{1/n})^{-1}$, and optimal functions are characteristic functions of balls [Federer-Fleming, '60]; (proved also by optimal transport).
- **Interpolation inequality.** For s, q s.t. $1 \leq q < s < 1^*$,

$$\|u\|_{L^s(\mathbb{R}^n)} \leq \|u\|_{L^q(\mathbb{R}^n)}^{1-\theta} \|u\|_{L^{1^*}(\mathbb{R}^n)}^{\theta}, \qquad (2)$$

  where $\frac{1}{s} = \frac{(1-\theta)}{q} + \frac{\theta}{1^*}$ i.e. $\theta = \frac{n(s-q)}{s(n-q(n-1))}$.
- Characteristic functions of balls are also optimal functions in the interpolation inequality (2).

Motivation

- **Sharp L^1 Gagliardo-Nirenberg inequalities.** Combining the L^1-Sobolev and the interpolation inequalities, we obtain the L^1 Gagliardo-Nirenberg inequality: for $1 \leq q < s < 1^*$,

$$\|u\|_{L^s(\mathbb{R}^n)} \leq \left(n\gamma_n^{1/n}\right)^{-\theta} \|\nabla u\|^{\theta}_{\mathcal{M}(\mathbb{R}^n)} \|u\|^{1-\theta}_{L^q(\mathbb{R}^n)} \quad \forall u \in D^{1,q}(\mathbb{R}^n) \tag{3}$$

where $D^{1,q}(\mathbb{R}^n) := \{u \in L^q(\mathbb{R}^n) : \|\nabla u\|_{\mathcal{M}(\mathbb{R}^n)} < \infty\}$.

- Characteristic functions of balls are optimal functions in (3) as they are optimal in both L^1 Sobolev and interpolation inequalities. Then the L^1-Gagliardo-Nirenberg inequality (3) is sharp with the best constant being $\left(n\gamma_n^{1/n}\right)^{-\theta}$.

Link between $(\mathcal{P})$ and L^1 Gagliardo-Nirenberg inequalities

By a scaling argument, the sharp L^1 Gagliardo-Nirenberg inequality (3) is related to problem $(\mathcal{P})$ when $F(t) = \frac{t^q}{q}$ and $G(t) = t^s$.

Proposition (A, 2008)

Let q and s be such that $1 \leq q < s < 1^*$. If the $(\mathcal{P})$-type problem

$$m_+ := \inf \left\{ E(u) := \int_{\mathbb{R}^n} d|\nabla u| + \frac{1}{q} \int_{\mathbb{R}^n} |u|^q : \ \|u\|_{L^s} = 1 \right\}$$

admits a minimizer u_∞, then the L^1 Gagliardo-Nirenberg inequality (3) holds, and the best constant K_{opt} and optimal functions are explicitly given in terms of the minimizer u_∞ as:

$K_{opt} = [K(n,q,s)/m_+]^{\frac{n+s-nq}{s[n-q(n-1))}}$, *where* $m_+ = E(u_\infty)$,

$K(n,q,s) = \frac{\alpha+\beta}{(q\alpha)^{\frac{\alpha}{\alpha+\beta}} \beta^{\frac{\beta}{\alpha+\beta}}}$, $\alpha = n - s(n-1)$, $\beta = n(s-q)$, *and*

$u_{\sigma,x_0}(x) = C u_\infty(\sigma(x - x_0))$, $C, \sigma \in \mathbb{R}$ *and* $x_0 \in \mathbb{R}^n$ *are arbitrary.*

Proof of the proposition

- u_∞ is a minimizer implies:

$$E(u_\infty) \leq E\left(\frac{u}{\|u\|_s}\right) = \frac{\|\nabla u\|_{\mathcal{M}}}{\|u\|_s} + \frac{\|u\|_q^q}{q\|u\|_s^q} \quad \forall u \in D^{1,q}(\mathbb{R}^n)$$

with equality if $u = u_\infty$.

- Scaling: $u_\lambda(x) = u\left(\frac{x}{\lambda}\right)$, $\lambda > 0$:

$$E(u_\infty) \leq \lambda^{n-1-\frac{n}{s}} \frac{\|\nabla u\|_{\mathcal{M}}}{\|u\|_s} + \lambda^{n(1-\frac{q}{s})} \frac{\|u\|_q^q}{\|u\|_s^q} := f(\lambda)$$

- Optimization in λ:

$$E(u_\infty) \leq \min_{\lambda>0} f(\lambda) = f(\lambda_{min})$$

gives the sharp L^1 Gagliardo-Nirenberg inequality, with u_∞ as an optimal function.

Goals

- Generalize the results on the particular problem $(\mathcal{P})$ of the proposition (i.e., extremality of characteristic functions of balls for $(\mathcal{P})$ when $F(t) = \frac{t^q}{q}$ and $G(t) = t^s$, $1 \leq q < s < 1^*$) to more general functions F and G.
- Derive other geometric inequalities involving the total variation with their best constants and optimal functions, by choosing different examples of functions F and G.

  Example. The sharp L^1 logarithmic Sobolev inequality is obtained by choosing $F(t) = t \ln t$ and $G(t)) = t$,

$$\int_{\mathbb{R}^n} |u| \ln \left(\frac{e^n \gamma_n |u|}{\|u\|_{L^1(\mathbb{R}^n)}} \right) \leq \int_{\mathbb{R}^n} d\,|\nabla u|, \quad \forall u \in \mathrm{BV}(\mathbb{R}^n).$$

- Prove existence and nonexistence results for some PDEs involving the 1-Laplacian operator $\Delta_1 u := \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)$; see [Demengel, '02], [Bellettini, Caselles, Novaga, '02], [Andreu, Caselles, Díaz, Mażon, '02].

The main result

Theorem

Assume that $F, \ G : [0,\infty) \to [0,\infty)$ *are continuous and increasing functions of class* C^1 *on* $(0,\infty)$ *s.t.* $F(0) = G(0) = 0$ *and* $\lim_{t\to\infty} G(t) = \infty$.

- *Then the infimum value* $m_\pm$,

$$m_\pm := \inf \left\{ E_\pm(u) = \int_{\mathbb{R}^n} d|\nabla u| \pm \int_{\mathbb{R}^n} F(|u|) : \int_{\mathbb{R}^n} G(|u|) = 1 \right\}$$

of problem $(\mathcal{P})$ *is given by the 1-dimensional variational problem*

$$m_\pm = \inf_{\alpha>0} H_\pm(\alpha)$$

where

$$H_\pm(\alpha) := n\gamma_n^{1/n} \frac{\alpha}{G(\alpha)^{(n-1)/n}} \pm \frac{F(\alpha)}{G(\alpha)}.$$

The main result

Theorem (Continuation)

- *If the infimum* $m_\pm$ *of* $H_\pm(\alpha)$ *is attained, that is, if the set* $V_\pm$ *of the minimizers of* $H_\pm(\alpha)$, $\alpha > 0$, *is non-empty,*

$$V_\pm := \{\alpha > 0,\ H_\pm(\alpha) = m_\pm\} \neq \emptyset,$$

then $(\mathcal{P})$ *has minimizers, which are characteristic functions of balls; more precisely the minimizers of* $(\mathcal{P})$ *are of the form*

$$\pm u_\alpha(x_0 + .) \quad \textit{with} \quad x_0 \in \mathbb{R}^n,\ \alpha \in V_\pm,$$

where

$$u_\alpha := \alpha \chi_{B_{\rho_\alpha}} \ \textit{and}\ \rho_\alpha := \frac{1}{\gamma_n^{1/n} G(\alpha)^{1/n}}.$$

- *On the contrary, if the infimum* $m_\pm$ *of* $H_\pm(\alpha)$ *is not attained in* $(0, \infty)$, *then* $(\mathcal{P})$ *has no minimizers.*

Proof of the main theorem - Claim 1

- **Claim 1. Symmetrization.** By symmetrization and a convenient change of variable $u \mapsto v$, we can rewrite problem $(\mathcal{P})$ as the minimization of a strictly concave functional $J_{\pm}(v)$ over a convex set K.
  More precisely, $(\mathcal{P})$ is equivalent to

  $$(\mathcal{P})_{\mathbf{v}} : \quad m_{\pm} = \inf\{J_{\pm}(v) : v \in K\}$$

  where

  $$J_{\pm(v)} := n\gamma_n \int_0^\infty v^{(n-1)/n}(t)dt \pm \gamma_n \int_0^\infty F'(t)v(t)dt$$

  is a strictly concave functional, and K is the set of nonincreasing real-valued functions $v : (0,\infty) \to (0,\infty)$ satisfying the linear constraint

  $$\int_0^\infty G'(t)v(t)dt = \frac{1}{\gamma_n}.$$

Proof of claim 1

- **Symmetrization.** Replacing u by its radially-symmetric decreasing rearrangement u^* will decrease the energy, $E_\pm(u^*) \leq E_\pm(u)$, but conserve the constraint, $\int_{\mathbb{R}^n} G(|u|) = \int_{\mathbb{R}^n} G(|u^*|)$; this follows from Pòlya-Szegö inequality and equimeasurability of rearrangement.
  Then the minimization in $(\mathcal{P})$ can be performed over the u^*.
- For such a u^*, the level set $\{u^* > t\}$ is a ball centered at the origin; call $\beta(t) > 0$ its radius. Then by the co-area formula [Fleming-Rishel, '60] and the layer cake representation (Cavalieri formula), we can rewrite $E_\pm(u^*)$ in terms of $\beta^n(t)$.
- It is more convenient to use the change of variable $v = \beta^n$, and write $E_\pm(u^*)$ in terms of v, so that $E_\pm(u^*) = J_\pm(v)$.
- With this change of variable, the constraint in $(\mathcal{P})$ becomes

$$1 = \int_{\mathbb{R}^n} G(|u^*|) = \gamma_n \int_0^\infty G'(t)v(t)dt.$$

Proof of the main theorem - Claim 2

- **Claim 2. Reduction to a 1-d problem.** The minimizer of $(\mathcal{P})_v$ - if it exists - is attained at a function of the form $v_\alpha := \frac{\chi_{[0,\alpha]}}{\gamma_n G(\alpha)}$ for some $\alpha > 0$ (i.e. an extreme point of K). Moreover, $J_\pm(v_\alpha) = H_\pm(\alpha)$, and $(\mathcal{P})_v$ reduces to the 1-d problem

$$m_\pm = \inf_{\alpha>0} J_\pm(v_\alpha) = \inf_{\alpha>0} H_\pm(\alpha)$$

- **Proof:**
  - $v_\alpha \in K$ implies $m_\pm \leq \inf_{\alpha>0} J_\pm(v_\alpha) = \inf_{\alpha>0} H_\pm(\alpha)$.
  - Conversely, any $v \in K$ can be represented in terms of the v_α as

$$v(t) = \int_0^\infty v_\alpha(t) d\mu_v(\alpha) \quad \text{where} \quad d\mu_v(\alpha) := -\gamma_n G(\alpha) v'(\alpha)$$

is a probability measure on $(0, \infty)$; so that by Jensen's inequality, we have for all $v \in K$

$$J_\pm(v) \geq \int_0^\infty J_\pm(v_\alpha) d\mu_v(\alpha) = \int_0^\infty H_\pm(\alpha) d\mu_v(\alpha) \geq \inf_{\alpha>0} H_\pm(\alpha).$$

Proof of the main theorem - Claim 3

- **Claim 3. Characterization of the minimizers.** The minimizers of $(\mathcal{P})$ - if they exist - are of the form $\pm u_\alpha(x_0 + .)$ where α is a minimizer of $H_\pm$, i.e. $H_\pm(\alpha) = m_\pm$, and $u_\alpha = \alpha\chi_{B_{\rho_\alpha}}$ with $\rho_\alpha = \frac{1}{\gamma_n^{1/n} G(\alpha)^{1/n}}$
- **Proof**
  - **Existence.** If $\alpha > 0$ is a minimizer of $H_\pm$, i.e. $H_\pm(\alpha) = m_\pm$, then the corresponding $u_\alpha(x)$ is admissible in $(\mathcal{P})$ and satisfies $m_\pm = H_\pm(\alpha) = J_\pm(v_\alpha) = E_\pm(u_\alpha)$; so u_α is a minimizer of $(\mathcal{P})$.
    And since $E_\pm(u)$ is invariant under translation and sign change, $u \mapsto -u(x_0+)$, then the functions $\pm u_\alpha(x_0 + .)$ are minimizers.
  - **Unique characterization.** If $\bar{u}$ is another minimizer of $(\mathcal{P})$, then there exists a minimizer α of $H_\pm$ s.t $\bar{u}$ differs from $\pm u_\alpha$ by a translation; this follows from rearrangement argument, strict concavity of $J_\pm$, and optimality in isoperimetric inequality.
- **Non-existence.** If $\inf_{\alpha>0} H_\pm(\alpha)$ is not attained in $(0, \infty)$, $\exists\, u_{\alpha_n}$ s.t. $\alpha_n \to 0$ or $\alpha_n \to \infty$; then $(\mathcal{P})$ has no minimizers.

- **Extensions.** The main result extends naturally to continuous functions F which are difference of two increasing functions F_1 and F_2 such that $F_1(0) = F_2(0) = 0$. Indeed, we can show that $E_\pm(u^*) = J_\pm(v)$ by decomposing $E_\pm(u^*)$ into terms of F_1 and F_2 and combine them back later; **e.g.** $F(t) = t \ln t$.

Proposition (sharp inequalities)

Assume F and G are such that $H_\pm(\alpha)$ has a minimizer α_∞, $H_\pm(\alpha_\infty) = m_\pm$, and consider the corresponding minimizer u_{α_∞} of $(\mathcal{P})$. Then the sharp inequality

$$\int_{\mathbb{R}^n} d|\nabla u| \pm \int_{\mathbb{R}^n} F(|u|) \leq m_\pm$$

holds for all functions u with finite total variation satisfying the constraint $\int_{\mathbb{R}^n} G(|u|) = 1$. Moreover, the optimal functions are $u = \pm u_{\alpha_\infty}(x_0 + .)$, $x_0 \in \mathbb{R}^n$.

Examples of sharp inequalities

- **Sharp L^1 logarithmic Sobolev inequality.** Choosing $F(t) = t \ln t$ and $G(t) = t$ in the proposition, $H_-(\alpha)$ is attained at $\alpha_\infty = 1/\gamma_n$, and $u_{\alpha_\infty} = \chi_{B_1}/\gamma_n$. We then have

$$\int_{\mathbb{R}^n} |u| \ln\left(\frac{e^n \gamma_n |u|}{\|u\|_{L^1(\mathbb{R}^n)}}\right) \leq \int_{\mathbb{R}^n} d\,|\nabla u|, \quad \forall u \in \mathrm{BV}(\mathbb{R}^n)$$

  and the optimal functions are $\pm u_{\alpha_\infty}(x_0 + .)$
- **Sharp L^1 Gagliardo-Nirenberg inequality.** Choose $F(t) = t^q/q$ and $G(t) = t^s$, with $1 < q < s < 1^*$ in the proposition. Then $H_+(\alpha)$ has a unique minimizer and we deduce the sharp L^1 Gagiardo-Nirenberg inequality.
- **Sharp L^1 Sobolev inequality.** Choose $F = 0$ and $G(t) = t^{1^*}$. Then $H(\alpha) = n\gamma^{1/n}$ has all $\alpha > 0$ as minimizers, and we deduce the sharp L^1 Sobolev inequality.

Proposition

If u is a non-negative minimzer of

$$\inf\left\{\int_{\mathbb{R}^n} d|\nabla u| + \int_{\mathbb{R}^n} F(|u|) : \int_{\mathbb{R}^n} G(|u|) = 1\right\}$$

then u satisfies the 1-Laplacian PDE (i.e. the Euler-Lagrange equation)

$$-\Delta_1 u + F'(u) = \lambda G'(u)$$

where λ is a Lagrange multiplier for the constraint $\int_{\mathbb{R}^n} G(u) = 1$.

Examples.

- Choosing $F(t) = t^q/q$ and $G(t) = t^s$ with $1^* < s < q$ or $1 < q < s < 1^*$, then $-\Delta_1 u = u^{s-1} - u^{q-1}$ has nontrivial nonnegative solutions in $D^{1,q}(\mathbb{R}^n)$.
- Similarly, if $s < q < 1 + \frac{s}{n}$, then $-\Delta_1 u = u^{s-1} + u^{q-1}$ has nontrivial nonnegative solutions in $\mathrm{BV}(\mathbb{R}^n)$.

A non existence result for 1 Laplacian PDEs

- Proposition

  In addition to the assumptions on F anf G in the main theorem, if F is convex and G is concave, then for any $\lambda \geq 0$, the PDE

$$\begin{cases} -\Delta_1 u = \lambda G'(u) - F'(u) \\ u \geq 0, \ \int_{\mathbb{R}^n} G(u) = 1 \end{cases} \tag{4}$$

  has no solutions.

  - **Proof.**
    - The convexity of F and concavity of G imply that u solves the PDE (4) iff it is a minimizer of the variational problem $(\mathcal{P})$.
    - But $H_+(\alpha)$ does not have a minimizer in $(0, \infty)$; in fact, $m_+ = 0$ as $\lim_{\alpha \to 0^+} H_+(\alpha) = 0$ but $H_+(\alpha) > 0 \ \ \forall \alpha > 0$. Therefore, problem $(\mathcal{P})$ has no minimizers, and so the PDE (4) has no solutions.
  - **Example.** Choosing $G(t) = t$ and $F(t) = t^q/q$, we have: If $q > 1$, the PDE $-\Delta_1 u = 1 - u^{q-1}$ has no solution in $D^{1,q}$.

Thank you!

Happy Birthday Terry!