Limiting normal approach for quasiconvex analysis

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Outline

Alternative title: A new step in quasiconvex calculus

I- Introduction

II- Limiting normal operator

- a- Definition
- b- Main properties

III- And what about previous concepts?

- A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *quasiconvex* on K if,

for all $x, y \in K$ and all $t \in [0,1]$,

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}.$$

or

for all $\lambda \in \mathbb{R}$, the sublevel set

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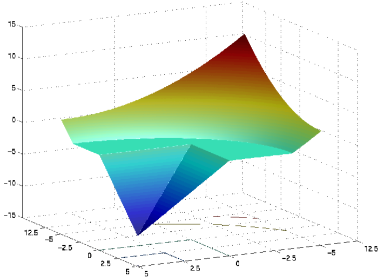
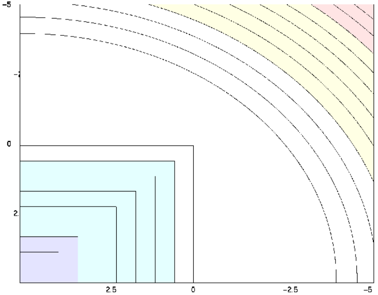
- f is quasiconvex iff ∂f is quasimonotone
  $$\text{iff } \exists\, x^* \in \partial f(x) \ : \ \langle x^*, y-x \rangle > 0$$
  $$\Rightarrow \forall\, y^* \in \partial f(y),\ \langle y^*, y-x \rangle \geq 0$$

Example





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- A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *semistrictly quasiconvex* on K if,

f is quasiconvex and for any $x, y \in K$,

$$f(x) < f(y) \Rightarrow f(z) < f(y), \quad \forall z \in [x, y[.$$

convex $\Rightarrow$ semistrictly quasiconvex $\Rightarrow$ quasiconvex

Motivations :
Our aim is to develop a “first order tool” for quasiconvex analysis/optimization that enjoy

- some generalized monotonicity
- some semicontinuity/closedness
- some sufficient optimality conditions (local or global)
- some calculus rules

Outline

Definition (Limiting sublevel set)

For a lower semicontinuous function $f : \mathbb{R}^m \to \mathbb{R}$ we define the *limiting sublevel set map* $S_f^l : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ as follows

$$S_f^l(\bar{x}) \equiv \operatorname*{Liminf}_{x \to \bar{x}} S_f(x), \quad \forall \bar{x} \in \mathbb{R}^m.$$

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Note the limiting sublevel set map S_f^l is closed-valued by definition. We further observe that the lower limit in the above definition may be restricted as follows.

Lemma

Let $f : \mathbb{R}^m \to \mathbb{R}$ be a lower semicontinuous function, then

$$S_f^l(\bar{x}) = \operatorname*{Liminf}_{\substack{x \longrightarrow \bar{x} \\ S_f(\bar{x})}} S_f(x).$$

As for the classical sublevel set S_f and strict sublevel set $S_f^<$, the convexity of the limit sublevel set characterizes the quasiconvexity of a lower semicontinuous function.

Lemma (Characterization of quasiconvexity in terms of S_f^l)

Let $f : \mathbb{R}^m \to \mathbb{R}$ be a lower semicontinuous function. Then f is quasiconvex if and only if $S_f^l(x)$ is convex for all $x \in \mathbb{R}^m$.

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Corollary (Inner semicontinuity of S_f^l)

For any lower semicontinuous function f, the limiting sublevel set map S_f^l is inner semicontinuous.

The limiting definition, inspired by the similar concept of limiting subdifferential, turns out to have the following very easy and natural equivalent explicit formulation for any lower semicontinuous function.

Theorem (Explicit formula for $S^l_f(x)$)

Let f be a lower semicontinuous function and $x \in \mathbb{R}^m$. Then

$$S^l_f(x) = \begin{cases} \bar{S}^<_f(x) & \text{if } x \in \bar{S}^<_f(x), \\ S_f(x) & \text{otherwise.} \end{cases}$$

In particular one always has $\bar{S}^<_f(x) \subset S^l_f(x) \subset S_f(x)$.

Definition (Limiting normal operator)

For quasiconvex lower semicontinuous function f the *limiting normal operator* is a set-valued map $N^l_f : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ defined as

$$N^l_f(x) = (S^l_f(x) - x)^o, \quad \forall x \in \mathbb{R}^m.$$

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Note that for a quasiconvex function one has $\left(S_f^l(x) - x\right)^o = \left(T_{S_f^l(x)}(x)\right)^o$. Thus, the above introduced limiting normal operator is a local notion. An alternative definition of the limiting normal operator can be given in terms of upper limit of normal operator.

From classical subdifferential calculus (Clarke's book, e.g.):

- *if f is Lipschitz on $\mathbb{R}^m$*
- *and $0 \notin \partial f(x)$*

then

$$cone\,(\partial f(x)) \subset N_f^l(x)$$

and if f is regular and semi-strictly quasiconvex,

$$cone\,(\partial f(x)) = N_f(x).$$

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and if f is regular and semi-strictly quasiconvex,

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Theorem (Relation of subdifferential and N^l_f)

For quasiconvex lower semicontinuous function $f : \mathbb{R}^m \to \mathbb{R}$, we have

$$[\overline{cone}\,(\partial f(\bar{x})) \cup \partial^\infty f(\bar{x})] \subset N^l_f(\bar{x}),$$

where equality holds provided $0 \notin \partial f(\bar{x})$.

Theorem

For any quasiconvex lower semicontinuous function f it holds

$$N_f^l(x) = \mathop{Limsup}_{x \to \bar{x}} N_f(x).$$

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Theorem

Let $f : \mathbb{R}^m \to \mathbb{R}$ be a lower semicontinuous quasiconvex function. Then

1. N_f^l *is quasimonotone,*
2. N_f^l *is outer semicontinuous.*

Now consider the minimization problem

$$\min f(x) \qquad \text{subject to} \qquad x \in K, \tag{1}$$

where $K \subset \mathbb{R}^m$ is nonempty subset of $\mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}$ is quasiconvex. Then, necessary and sufficient optimality conditions may be stated as follows.

Theorem (Necessary optimality conditions)

Let $f : \mathbb{R}^m \to \mathbb{R}$ be a lower semicontinuous quasiconvex function, K be a nonempty convex set and $\bar{x} \in K$ be a solution to (1) such that $\bar{x} \in \bar{S}_f^<(\bar{x})$. Then it holds

$$0 \in N_f^l(\bar{x}) \setminus \{0\} + N_K(\bar{x}).$$

Theorem

Let $f : \mathbb{R}^m \to \mathbb{R}$ be a continuous quasiconvex function, K be a nonempty subset of $\mathbb{R}^m$ and $\bar{x} \in K$. Then $\bar{x}$ a is local solution to (1) if one of the following hypothesis is satified:

1. *the point $\bar{x}$ is a solution of the Stampacchia variational inequality defined by $N_f^l(\cdot) \setminus \{0\}$ and K;*
2. *the set K is convex and $0 \in N_f^l(\bar{x}) \setminus \{0\} + N_K(\bar{x})$.*

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Back to (local) paradise.....

Corollary

Let $f : \mathbb{R}^m \to \mathbb{R}$ be a continuous quasiconvex function, K be a nonempty convex set and $\bar{x} \in K$ be such that $\bar{x} \in \bar{S}_f^<(\bar{x})$. Then

$$\bar{x} \in \arg\min_K f \quad \Leftrightarrow \quad 0 \in N_f^l(\bar{x}) \setminus \{0\} + N_K(\bar{x}).$$

Corollary

Let $f : \mathbb{R}^m \to \mathbb{R}$ be a continuous quasiconvex function, K be a nonempty convex set and $\bar{x} \in K$ be such that $\bar{x} \in \bar{S}_f^<(\bar{x})$. Then

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Back to quasi-paradise.....

For quasiconvex functions there is two stable operations:

- Let $\mathcal{F} = \{f_i : \mathbb{R}^m \to \mathbb{R}\}_{i \in I}$ be a finite family of continuous quasiconvex functions and define

$$g(x) \equiv \max_{i \in I} f_i(x).$$

  Clearly g is finite, quasiconvex and continuous on $\mathbb{R}^m$.

- For any function $g : \mathbb{R}^m \to \mathbb{R}$ defined as $g(x) = \theta(f(x))$ where f is a lower semicontinuous quasiconvex function and $\theta : \mathbb{R} \to \mathbb{R}$ is an non-decreasing function

For any x, the index set $I(x)$ of active functions stands for $I(x) \equiv \{i \in I : f_i(x) = g(x)\}$.

Theorem (Limiting normal operator to maximum)

Let $\mathcal{F} = \{f_i : \mathbb{R}^m \to \mathbb{R}\}_{1 \in I}$ be a finite family of continuous quasiconvex functions and g be defined as above. Then

$$N^l_g(\bar{x}) \subset \sum_{i \in I(\bar{x})} N^l_{f_i}(\bar{x})$$

if the following constraint qualification is satisfied at $\bar{x} \in \mathbb{R}^m$

$$\forall\, i \in I(\bar{x}) v_i \in N^l_{f_i}(\bar{x}) \quad \text{and} \quad \sum_{i \in I(\bar{x})} v_i = 0 \quad \Longrightarrow \quad \forall\, i \in I(\bar{x}) v_i = 0. \quad (2)$$

Let us recall that condition (2) means that the convex sets $\{S^l_{f_i}(\bar{x})\}_{i \in I(\bar{x})}$ can not be separated. Equivalently we may say that $\bar{x}$ is not an extremal point of the system $\{S^l_{f_i}(\bar{x})\}_{i \in I(\bar{x})}$, see Mordukhovich[Corollary 2.4 and Theorem 2.8]Mor06.

For any function $g : \mathbb{R}^m \to \mathbb{R}$ defined as $g(x) = \theta(f(x))$ where f is a lower semicontinuous quasiconvex function and $\theta : \mathbb{R} \to \mathbb{R}$ is an increasing function then $N^l_g(x) = N^<_f(x)$ for any $x \in \mathbb{R}^m$ as could be easily verified from definitions. For the case of the composition with a (only) non-decreasing function the chain rule is as follows.

Theorem (Chain rule for limiting normal operator)

Consider a lower semicontinuous quasiconvex function $f : \mathbb{R}^m \to \mathbb{R}$, a non-decreasing lower semicontinuous function $\theta : \mathbb{R} \to \mathbb{R}$ and their lower semicontinuous quasiconvex composition $g(x) = \theta(f(x))$. Then for any $\bar{x} \in \mathbb{R}^m$, the limiting normal operator $N^l_g(\bar{x})$ and strict normal operator $N^<_g(\bar{x})$ at point $\bar{x}$ satisfy

$$
\begin{aligned}
N^l_g(\bar{x}) &\subset N^l_f(\bar{x}) \\
N^<_g(\bar{x}) &\supset N^<_f(\bar{x})
\end{aligned}
$$

These inclusions become equalities, that is $N^l_g(\bar{x}) = N^l_f(\bar{x})$, provided $\bar{x} \in \bar{S}^<_g(\bar{x})$.

Outline

Sublevel set:

$$S_\lambda = \{x \in X \ : \ f(x) \leq \lambda\}$$
$$S_\lambda^> = \{x \in X \ : \ f(x) < \lambda\}$$

Normal operator:

Define $N_f(x) : X \to 2^{X^*}$ by

$$\begin{aligned} N_f(x) &= N(S_{f(x)}, x) \\ &= \{x^* \in X^* \ : \ \langle x^*, y - x\rangle \leq 0, \ \forall y \in S_{f(x)}\}. \end{aligned}$$

With the corresponding definition for $N_f^>(x)$

Seminal work of Borde-Crouzeix (1980)

Adjusted sublevel set:

For any $x \in \operatorname{dom} f$, we define

$$S_f^a(x) = S_{f(x)} \cap \overline{B}(S_{f(x)}^<, \rho_x)$$

where $\rho_x = dist(x, S_{f(x)}^<)$, if $S_{f(x)}^< \neq \emptyset$.

Ajusted normal operator:

$$N_f^a(x) = \{x^* \in X^* \ : \ \langle x^*, y - x \rangle \leq 0, \ \forall y \in S_f^a(x)\}$$

defined in D.A.-Hadjisavvas SIOPT (2005)

Final assessment

Operator	Quasimono.	Closedness	Suff.Opt. cond.	Calculus rules
N_f	Yes	No	No	
$N_f^<$	No	Yes	No	
N_f^a	Yes	Yes	Yes	
N_f^l	Yes	Yes	Yes	Yes

References

- J. Borde and J.-P. Crouzeix, *Continuity properties of the normal cone to the level sets of a quasiconvex function*, JOTA (1990), 66:415–429.
- D. A. & N. Hadjisavvas, *Adjusted sublevel sets, normal operator and quasiconvex programming*, SIAM J. Optim., 16 (2005), 358367.
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- D. Aussel, *New developments in Quasiconvex optimization* in *Fixed Point Theory, Variational Analysis, and Optimization*, Ed Taylor & Francis, 2014, 173-208.
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- A. Cabot and L. Thibault, *Sequential formulae for the normal cone to sublevel sets*, Trans. AMS (2014), 366:6591–6628.

But ...

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- $N_f^>(x) = N(S_{f(x)}^>, x)$ has no quasimonotonicity properties

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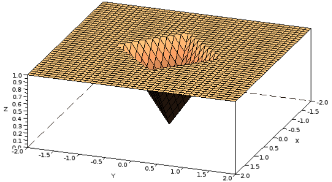
Example

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(a,b) = \begin{cases} |a| + |b|, & \text{if } |a| + |b| \leq 1 \\ 1, & \text{if } |a| + |b| > 1 \end{cases}.$$



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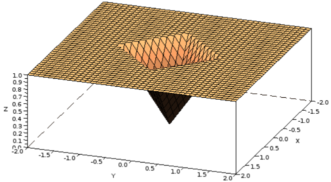

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Consider $x = (10,0)$, $x^* = (1,2)$, $y = (0,10)$ and $y^* = (2,1)$.

We see that $x^* \in N^<(x)$ and $y^* \in N^<(y)$ (since $|a| + |b| < 1$ implies $(1,2) \cdot (a-10, b) \le 0$ and $(2,1) \cdot (a, b-10) \le 0$)

while $\langle x^*, y - x\rangle > 0$ and $\langle y^*, y - x\rangle < 0$. Hence $N^<$ is not quasimonotone.

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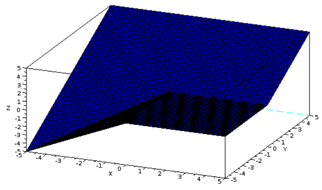

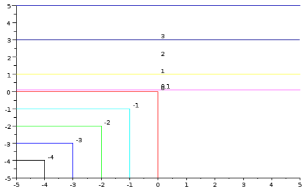

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- $N_f^{>}(x) = N(S^{>}_{f(x)}, x)$ has no quasimonotonicity properties

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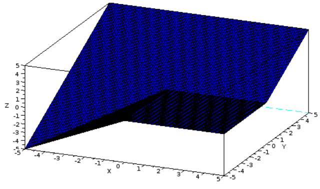

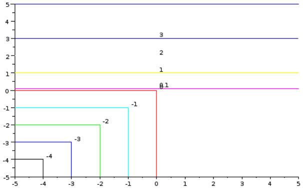

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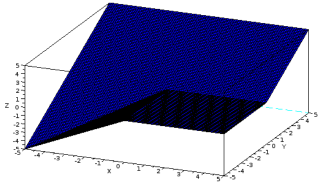

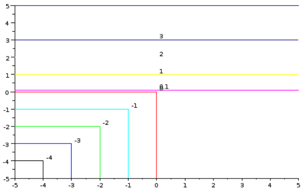

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These two operators are essentially adapted to the class of semi-strictly quasiconvex functions. Indeed in this case, for each $x \in \operatorname{dom} f \setminus \arg\min f$, the sets $S_{f(x)}$ and $S^{<}_{f(x)}$ have the same closure