Optimal pits and optimal transportation

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CESAME Seminar in Systems and Control, UCL
November 18, 2014

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To dig a hole in the ground and excavate valuable minerals

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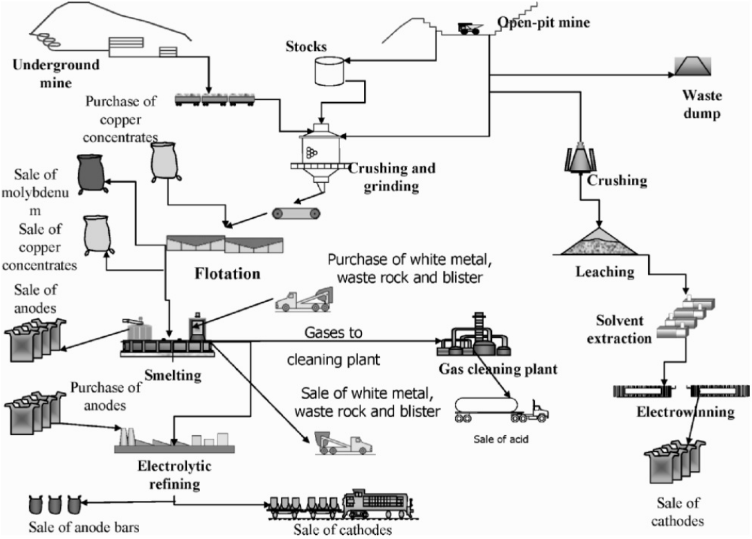
Chuquicamata copper mine, Chile
(4.3 km $\times$ 3 km $\times$ 900 m)

Mining Processes



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4. Execution. . .

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Bingham Canyon copper mine, Utah
(massive landslide, 10 April 2013)

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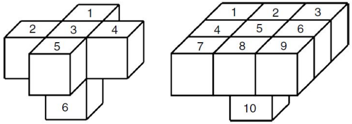

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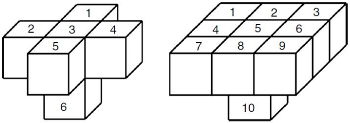

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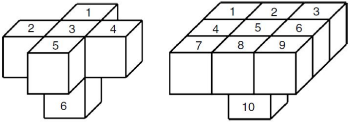

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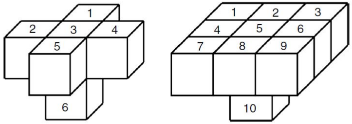

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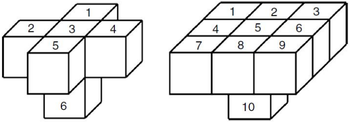

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- implemented in commercial software (Whittle, Geovia)

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All these continuous space approaches suffer from lack of convexity

- how to deal with *local optima?*

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  - $g(x)dx$ net profit from volume element $dx = dx_1\, dx_2\, dx_3$ at x
  - $g(F) := \int_F g(x)dx$ total net profit from pit F
  - assume $\int_E \max\{0, g(x)\}\, dx > 0$ (there is some profit to be made)

Optimum pit problem: find $F^* \in \arg\max\{g(F) : F \text{ is a pit}\}$

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A Profit Allocation Model

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  These restrictions will be modelled by a “transportation” (or allocation) cost function $c : X \times Y \longrightarrow \mathbb{R}$

Allocation “Costs” and Optimum Profit Allocation

X	Y	$c(x,y)$
$x \in E^+$	$y \in \Gamma(x)$	0
$x \in E^+$	$y \notin \Gamma(x),\ y \in E^-$	$+\infty$
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$$\min_{\pi} \mathrm{E}^{\pi}[c] := \int_{X \times Y} c(x,y)\,d\pi \quad \text{s.t. } \pi \in \Pi(\mu,\nu) \qquad \text{(K)}$$

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Proof: The set of positive Radon measures on compact space $X \times Y$ is weak-* compact, and the map $\pi \to \mathrm{E}^{\pi}[c]$ is weak-* l.s.c. □

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The Kantorovich Dual

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Potentials (duals, Lagrange multipliers)

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- there is no *duality gap* (in continuous variables)

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- i.e., transportation problem (K) is a *weak dual* to the optimum pit problem (P)

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c-Fenchel Conjugates

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Given $c : X \times Y \to \mathbb{R}$, define the c-Fenchel conjugates (or c-Fenchel-Legendre transforms)

- $p^\sharp : Y \to \mathbb{R}$ of any function $p \in L^1(X, \mu)$ by

$$p^\sharp(y) := \operatorname*{ess\,sup}_{x \in X} \left(p(x) - c(x, y)\right)$$

- $q^\flat : X \to \mathbb{R}$ of any function $q \in L^1(Y, \nu)$ by

$$q^\flat(x) := \operatorname*{ess\,inf}_{y \in Y} \left(q(y) + c(x, y)\right)$$

where $\operatorname{ess\,sup} f(x) = \inf_{N \in \mathcal{N}} \sup_{x \in X \setminus N} f(x)$, where $\mathcal{N}$ is the set of measurable subsets $N \subset X$ with $\mu(N) = 0$

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- To simplify, we'll write $\sup$ and $\inf$ instead of $\operatorname{ess\,sup}$ and $\operatorname{ess\,inf}$
- Similarly, all equalities and inequalities will be μ-a.e. in X and ν-a.e. in Y

Properties of c-Fenchel Conjugates

[Carlier, 2003; Ekeland, 2010]

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For all $x \in X$, $y \in Y$,

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Monotonicity:

$$p_1 \leq p_2 \implies p_1^\sharp \leq p_2^\sharp$$
$$q_1 \leq q_2 \implies q_1^\flat \leq q_2^\flat$$

c-Fenchel Transforms for the Open Pit Dual Problem

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$$p^\sharp(y) := \max\left\{p(\alpha), \sup_{x\,:\,y\in\Gamma(x)} p(x)\right\} \qquad \text{for } y \in E^-$$

$$p^\sharp(\omega) := \max\left\{p(\alpha), \sup_{x\in E^+} p(x) - 1\right\}$$

$$q^\flat(x) := \min\left\{1 + q(\omega), \inf_{y\in\Gamma(x)} q(y)\right\} \qquad \text{for } x \in E^+$$

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$p^\sharp$ and $q^\flat$ are increasing with respect to Γ:

$$x' \in \Gamma(x) \implies q^\flat\left(x'\right) \geq q^\flat(x)$$

$$y' \in \Gamma(y) \implies p^\sharp\left(y'\right) \geq p^\sharp(y)$$

c-Fenchel Transforms for the Open Pit Dual Problem

$$p^\sharp(y) := \max\left\{p(\alpha),\ \sup_{x\,:\,y\in\Gamma(x)} p(x)\right\} \qquad \text{for } y \in E^-$$

$$p^\sharp(\omega) := \max\left\{p(\alpha),\ \sup_{x\in E^+} p(x) - 1\right\}$$

$$q^\flat(x) := \min\left\{1 + q(\omega),\ \inf_{y\in\Gamma(x)} q(y)\right\} \qquad \text{for } x \in E^+$$

$$q^\flat(\alpha) := \min\left\{q(\omega),\ \inf_{y\in E^-} q(y)\right\}$$

$p^\sharp$ and $q^\flat$ are increasing with respect to Γ:

$$x' \in \Gamma(x) \implies q^\flat\left(x'\right) \geq q^\flat(x)$$
$$y' \in \Gamma(y) \implies p^\sharp\left(y'\right) \geq p^\sharp(y)$$

For a pit F, $\quad p_F = q_F^\flat$ and $q_F = p_F^\sharp$

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Translation Invariance

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Given $(p, q) \in \mathcal{A}$ and constants p_0, p_1, q_0, q_1 satisfying:

$$\mu\left(E^{+}\right)\left(q_0 - p_1\right) - \nu\left(E^{-}\right)\left(p_0 - q_1\right) = 0$$

define $\tilde{p}$ and $\tilde{q}$ by:

$$\begin{aligned}
\tilde{p}(\alpha) &= p(\alpha) - p_0 \\
\tilde{p}(x) &= p(x) - p_1 \quad \text{for} \quad x \in E^{+} \\
\tilde{q}(\omega) &= q(\omega) - q_0 \\
\tilde{q}(y) &= q(y) - q_1 \quad \text{for} \quad y \in E^{-}
\end{aligned}$$

Then:

$$J\left(\tilde{p}, \tilde{q}\right) = J(p, q)$$

c-Fenchel Transforms Give Local Improvements

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If $(p, q) \in \mathcal{A}$, then $p(x) - q(y) \leq c(x, y)$ for all (x, y), so that:

$$p(x) \leq \inf_y \{c(x, y) + q(y)\} = q^\flat(x)$$

$$q(y) \geq \sup_x \{p(x) - c(x, y)\} = p^\sharp(y)$$

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Therefore

$$\left(p, p^\sharp\right) \in \mathcal{A} \quad \text{and} \quad J\left(p, p^\sharp\right) \ge J(p,q)$$

$$\left(q^\flat, q\right) \in \mathcal{A} \quad \text{and} \quad J\left(q^\flat, q\right) \ge J(p,q)$$

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This implies $\qquad J(p, q) \leq J\left(p, p^\sharp\right) \leq J\left(p^{\sharp\flat}, p^\sharp\right)$

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This implies $\quad J(p, q) \le J\left(p, p^\sharp\right) \le J\left(p^{\sharp\flat}, p^\sharp\right)$

Letting $\bar{p} := p^{\sharp\flat}$ and $\bar{q} := p^\sharp$, we get:

$$J(p, q) \le J(\bar{p}, \bar{q})$$

$$\bar{p} = \bar{q}^\flat \text{ and } \bar{q} = \bar{p}^\sharp$$

A Dual Solution

A Dual Solution

Proposition 2: *Problem* (D) *has a solution* $(\bar{p}, \bar{q})$ *with*

$$\begin{array}{lll} \bar{p} = \bar{q}^\flat & 0 \leq \bar{p} \leq 1 & \bar{p}(\alpha) = 0 \\ \bar{q} = \bar{p}^\sharp & 0 \leq \bar{q} \leq 1 & \bar{q}(\omega) = 0 \end{array}$$

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- $\mathcal{A}$ convex closed in $L^1(\mu) \times L^1(\nu)$ is weakly closed, so $(\bar{p}, \bar{q}) \in \mathcal{A}$
- Since J is linear and continuous on $L^1(\mu) \times L^1(\nu)$, we get:
  $J(\bar{p}, \bar{q}) = \lim_n J(p_n, q_n) = \sup(\text{D})$ □

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Complementary Slackness, and Monotonicity

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If π is optimal to problem (K) and (p, q) to its dual (D), then

$$0 = J(p,q) - \int_{X\times Y} c(x,y)d\pi = \int_{X\times Y} \big(p(x) - q(y) - c(x,y)\big)d\pi$$

implying the CS conditions: $p(x) - q(y) - c(x,y) = 0, \ \ \pi$–a.e.

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$$y'' \succsim y' \succsim x'' \succsim x' \implies \bar{q}\left(y''\right) \geq \bar{q}\left(y'\right) \geq \bar{p}(x'') \geq \bar{p}\left(x'\right)$$

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Proof: The first and last inequalities follow from $\bar{q} = \bar{p}^{\sharp}$, $\bar{p} = \bar{q}^{\flat}$, and c-Fenchel conjugates increasing w.r.t. Γ

- the middle inequality follows from

$$\bar{p}^{\sharp}(y) = \max\left\{\bar{p}(\alpha), \sup_{x\,:\,y\in\Gamma(x)} \bar{p}(x)\right\} \quad \text{for all } y \in E^{-} \qquad \square$$

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$$F := \{x \mid \bar{p}(x) = 1\} \cup \{y \mid \bar{q}(y) = 1\}$$

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$$g(F) = \int_{F^+} d\mu - \int_{F^-} d\nu \leq \sup(\mathsf{P})$$

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- Let $G^+ := E^+ \backslash F^+$ and $G^- := E^- \backslash F^-$:
  since $\bar{p} = 1$ on F^+, $\bar{q} = 1$ on F^-, and $\bar{p}(\alpha) = \bar{q}(\omega) = 0$,

$$J(\bar{p}, \bar{q}) = \int_{F^+} d\mu - \int_{F^-} d\nu + \int_{G^+} \bar{p}\, d\mu - \int_{G^-} \bar{q}\, d\nu$$

Proof, continued

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$$\implies \quad J(\bar{p}, \bar{q}) = \int_{F^+} d\mu - \int_{F^-} d\nu = g(F)$$

- Hence $\quad g(F) = J(\bar{p}, \bar{q}) = \sup(\mathsf{D}) = \inf(\mathsf{K}) \ge \sup(\mathsf{P}) \ge g(F) \qquad \square$

Main Result

Theorem: *If*

- E *is compact,*
- Γ *is reflexive, transitive and has a closed graph, and*
- $g(x)$ *is continuous with* $\int_E \max\{0, g(x)\}\, dx > 0$,

then:

1. *Problem (P) has an optimum solution, i.e., an optimal pit* F
2. *Its indicator functions* (p_F, q_F) *define optimum potentials, i.e., optimal solutions to (D)*
3. *Problem (K) has an optimum solution (profit allocation) and is a strong dual to (P), i.e.,* $\min(K) = \max(P)$
4. *A pit* F *is optimal iff there exists a feasible solution* π *to (K) such that* (p_F, q_F) *satisfies the CS conditions*

Uniqueness?

Theorem [Matheron, 1975; also Topkis, 1976]:

1. *The family $\mathcal{F}$ of all pits is closed under arbitrary unions and intersections:*

$$\bigcup_{F \in \mathcal{G}} F \in \mathcal{F} \quad \textit{and} \quad \bigcap_{F \in \mathcal{G}} F \in \mathcal{F} \quad \textit{for all } \mathcal{G} \subseteq \mathcal{F}$$

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That's it, folks.

Any questions?