Conjugate Duality

and Linear Dynamics with Constraints or Uncertainty

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TerryFest, Limoges 2015

Conjugate Duality and Optimal Control

Conjugate convex functions in optimal control and the calculus of variations, RTR, 70
Generalized Hamiltonian equations for convex problems of Lagrange, RTR, 70

As envisioned in

Conjugate Duality and Optimization, RTR, 74

conjugate duality ideas led to advances in the theory of fully convex calculus of variations and optimal control problems.

Dual problems of Lagrange for arcs of bounded variation,, RTR, 76
Linear-quadratic programming and optimal control, , RTR, 87
Hamiltonian trajectories and duality in the optimal control of linear systems with convex costs,, RTR, 89
Convexity in Hamilton-Jacobi theory I: dynamics and duality, RTR & Wolenski, 01
Duality and uniqueness of convex solutions to stationary Hamilton-Jacobi equations, G., 05
Duality in convex problems of Bolza over functions of bounded variation, Pennanen & Perkkiö, 14

PS. Literature sample limited to Terry and his scientific descendants.

1. Pair a generalized problem of Bolza

$$\min \qquad l(x(0), x(T)) + \int_0^T L\left(x(t), \dot{x}(t)\right)\, dt,$$

where $L, l : \mathbb{R}^{2n} \to (-\infty, \infty]$ are convex with

$$\min \qquad l^*(p(0), -p(T)) + \int_0^T L^*\left(\dot{p}(t), p(t)\right)\, dt.$$

2. The (maximized) Hamiltonian

$$H(x,p) = \sup_{v \in \mathbb{R}^n} \left\{ v \cdot p - L(x,v) \right\}.$$

is convex in p (as usual) and concave in x.

3. Optimal control problems with linear dynamics and convex penalties (Linear-Convex Regulator) and dual optimal control problems with dual linear dynamics:

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \qquad\qquad \begin{bmatrix} -\dot{p} \\ q \end{bmatrix} = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} \begin{bmatrix} p \\ w \end{bmatrix}.$$

Obtain: existence and regularity of minimizers, necessary and sufficient Hamiltonian optimality conditions, conjugacy and Hamilton-Jacobi characterization of primal and dual value functions, regularity of the optimal value functions and of the optimal feedback, etc.

To make Terry feel (even more) at home

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$$f^*(p) = \sup_{x \in \mathbb{R}^n} \{x \cdot p - f(x)\}$$

Lyapunov inequalities for dual linear differential inclusions.

An elementary observation

Let $f : \mathbb{R}^n \to [0, \infty)$ be homogeneous of degree 2. Let A be a $n \times n$ matrix. Let $\gamma > 0$. If

$$f(Ax) \leq \gamma f(x) \quad \forall x \in \mathbb{R}^n$$

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then

$$\begin{aligned} f^*(A^T p) &= \sup_{x \in \mathbb{R}^n} \left\{ x \cdot A^T p - f(x) \right\} \\ &\leq \sup_{x \in \mathbb{R}^n} \left\{ x \cdot A^T p - \frac{1}{\gamma} f(Ax) \right\} \end{aligned}$$

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f^*(A^T p) &= \sup_{x\in\mathbb{R}^n} \left\{ x \cdot A^T p - f(x) \right\} \\
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\end{aligned}$$

Hence,

$$f^*(A^T p) \leq \gamma f^*(p) \quad \forall p \in \mathbb{R}^n.$$

In summary, if $f : \mathbb{R}^n \to [0, \infty)$ is convex and homogeneous of degree 2, then

$$f(Ax) \leq \gamma f(x) \quad \forall x \in \mathbb{R}^n \qquad \Longleftrightarrow \qquad f^*(A^T p) \leq \gamma f^*(p) \quad \forall p \in \mathbb{R}^n.$$

Similarly, subject to further positive-definiteness and differentiability assumption,

$$\nabla f(x) \cdot Ax < 0 \quad \forall x \neq 0 \qquad \Longleftrightarrow \qquad \nabla f^*(p) \cdot A^T p < 0 \quad \forall p \neq 0$$

follows from ∇f, ∇f^* being inverses of one another.

Background on linear differential inclusions

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The general problem of the stability of motion, Lyapunov, 78 B.C.A.

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- Asymptotic stability for arbitrary switching between $\dot{x} = A_i x$ $\iff$ asymptotic stability for linear differential inclusion

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- Asymptotic stability for (1) $\iff$ exists a convex and smooth Lyapunov function:

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Criteria of asymptotic stability of differential and difference inclusions ..., Molchanov, Pyatnitskiy, 17 A.C.A.

A converse Lyapunov theorem for a class of dynamical systems which undergo switching, Dayawansa, Martin, 29 A.C.A.

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- Asymptotic stability for (1) $\iff$ asymptotic stability for

$$\dot{p} \in \operatorname{conv} \left\{A_1^T p, A_2^T p, \ldots, A_m^T p\right\} \tag{2}$$

Stability of inclusions of linear type, Barabanov, 25 A.C.A.

Theorem (G. et al 06)

Suppose $V, V^* : \mathbb{R}^n \to \mathbb{R}$ are convex, differentiable, positive definite, and positively homogeneous of degree 2. Then the following are equivalent:

- *V is a Lyapunov function for the linear differential inclusion (1).*
- *V^* is a Lyapunov function for the dual linear differential inclusion (2).*

Conjugate convex Lyapunov functions for dual linear differential inclusions, G. et al., IEEE TAC, 06

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Conjugate convex Lyapunov functions for dual linear differential inclusions, G. et al., IEEE TAC, 06

Note:

- Extends to V, V^* homogeneous of degree p, q, where $1/p + 1/q = 1$.
- Extends to V, V^* homogeneous of degree 1, through polarity.
- Carries over to discrete time.
- If needed, both V and V^* can be simultaneously smoothed.

Theorem (G. 00)

If $V : \mathbb{R}^n \to [0, \infty)$ is proper, lsc, convex, then

$$V_\lambda(x) = (1-\lambda)^2 e_\lambda V(x) + \frac{\lambda}{2}\|x\|^2, \qquad \text{where } e_\lambda V(x) = \inf_{y \in \mathbb{R}^n} \left\{ V(y) + \frac{1}{2\lambda}\|y - x\|^2 \right\}$$

is differentiable, $V_\lambda \to V$ as $\lambda \searrow 0$, and $(V_\lambda)^* = (V^*)_\lambda$.

Self-dual smoothing of convex and saddle functions , G., JCA, 08

Lyapunov inequalities for convex processes.

Background on linear systems and convex processes

The linear control system

$$\dot{x} = Ax + Bu$$

is controllable (stabilizable).

$\iff$

The linear system

$$\dot{p} = A^T p, \quad q = B^T p.$$

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How about (nonnegativity, conical) constraints?

- Early work on positive controls.

Controllability in linear autonomous systems with positive controllers, Brammer 72

Global controllability of linear systems with positive controls, Saperstone 73

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Controllability of convex processes, Aubin, Frankowska, Olech 86

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Crash course (for details, consult *Convex Analysis*, Rockafellar, 70):

- Convex process: set-valued mapping the graph of which is a convex cone.
- The adjoint process F^* of a convex process F is:

$$(p, w) \in \operatorname{gph} F^* \;\equiv\; (-w, p) \in (\operatorname{gph} F)^*$$

- Example:

$$F(x) = \begin{cases} Ax + K & \text{if} \quad x \in X \\ \emptyset & \text{if} \quad x \notin X \end{cases} \qquad F^*(p) = \begin{cases} A^T p - X^* & \text{if} \quad p \in K^* \\ \emptyset & \text{if} \quad p \notin K^* \end{cases}$$

Theorem (G. 13)

Let $V, V^* : \mathbb{R}^n \to \mathbb{R}$ be convex, differentiable, positive definite, and positively homogeneous of degree 2. Suppose that the convex process $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is strict and closed. Let $\gamma > 0$. TFAE:

- *V is a weak Lyapunov function for $\dot{x} \in F(x)$:*

$$\forall x \in \mathbb{R}^n \; \exists v \in F(x) \qquad \nabla V(x) \cdot v \leq -\gamma V(x).$$

- *V^* is a Lypaunov function for $\dot{p} \in F^*(p)$:*

$$\forall p \in \mathrm{dom} F^* \; \forall w \in F^*(p) \qquad \nabla V^*(p) \cdot w \leq -\gamma V^*(p).$$

Lyapunov functions and duality for convex processes, G., SICON, 13

Note:

- Weak Lyapunov function:
$$V(x) = \min \left\{ \int_0^\infty \|\phi\|^2 + \|\dot{\phi}\|^2 \, dt \mid \phi(0) = x, \; \dot{\phi} \in F(\phi) \right\}.$$
- Carries over to discrete time.
- If needed, both V and V^* can be simultaneously smoothed.

Let A, B be matrices and U a closed convex cone. The following are equivalent:

- The linear system
$$\dot{x} = Ax + Bu \tag{3}$$
is asymptotically controllable with controls $u \in U$.
- The dual linear system
$$\dot{p} = A^T p, \ q = B^T p \tag{4}$$
is detectable through output $q \in U^*$.

Follows from *Introduction to the theory of differential inclusions*, Smirnov, 06.

Corollary (G. 14)

If $V, V^* : \mathbb{R}^n \to \mathbb{R}$ are convex, differentiable, positive definite, and positively homogeneous of degree 2, then the following are equivalent:

- *V is a control Lyapunov function for (3) with the control constraint $u \in U$,*
- *V^* is a Lypaunov function verifying detectability of (4) through U^*.*

Linear systems with conical constraints and convex Lyapunov functions in the framework of convex processes, G., IEEE CDC, 14

Dissipativity inequalities with convex storage and saddle supply functions.

Crash course (for details, consult *Convex Analysis*, Rockafellar, 70):

- For a proper closed saddle function $h : \mathbb{R}^m \times \mathbb{R}^n \to [-\infty, \infty]$ (convex in first variable, concave in second), the saddle conjugate class $[h^*]$ consists of proper closed saddle functions between

$$\begin{aligned} \underline{h^*}(p,q) &= \sup_{x\in\mathbb{R}^n} \inf_{y\in\mathbb{R}^m} \{p\cdot x + q\cdot y - h(x,y)\} \\ \overline{h^*}(p,q) &= \inf_{y\in\mathbb{R}^m} \sup_{x\in\mathbb{R}^n} \{p\cdot x + q\cdot y - h(x,y)\} \end{aligned}$$

- $([h^*])^* = [h]$
- Example: for $Q = Q^T \geq 0$, $R = R^T \geq 0$ and any S, if

$$h(x,y) = \frac{1}{2}x\cdot Qx - \frac{1}{2}y\cdot Ry + x\cdot Sy$$

and $M = \begin{bmatrix} Q & S \\ S^T & -R \end{bmatrix}$ is invertible then $h^*(p,q) = \frac{1}{2}\begin{bmatrix} p \\ q \end{bmatrix}^T M^{-1} \begin{bmatrix} p \\ q \end{bmatrix}$

- In particular: $\left(\frac{1}{2}x^2 - \frac{1}{2}y^2\right)^* = \frac{1}{2}p^2 - \frac{1}{2}q^2$, $(xy)^* = pq$.

Theorem (G. et al 04)

Let A, B, C, D be matrices. Let $V, V^* : \mathbb{R}^n \to \mathbb{R}$ be convex, positive definite, and homogeneous of degree 2. Let h, h^* be saddle functions, homogeneous of degree 2. TFAE:

(a) *for all x, d,*

$$\partial V(x) \cdot (Ax + Bd) \le -\gamma V(x) - h(Cx + Dd, d), \tag{5}$$

(b) *for all p, w,*

$$\partial V^*(p) \cdot (A^T p + C^T w) \le -\gamma V^*(p) + h^*(w, -B^T p - D^T w). \tag{6}$$

Dissipativity for dual linear differential inclusions through conjugate storage functions, G. et al., IEEE CDC, 04

- Inequalities (5), (6) characterize dissipativity properties of linear differential inclusions and their duals:

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \mathrm{conv}\left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix}_i \right\}_{i=1}^m \begin{bmatrix} x \\ d \end{bmatrix} \qquad \begin{bmatrix} \dot{\xi} \\ z \end{bmatrix} = \mathrm{conv}\left\{ \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix}_i \right\}_{i=1}^m \begin{bmatrix} \xi \\ w \end{bmatrix}.$$

- $h(c, d) = -c \cdot d$ relates to passivity, $h(c, d) = c^2 - d^2$ relates to finite $\mathcal{L}^2$-gain, etc.
- $h(c, d) = -\delta_0(d)$, $h^*(w, z) = \delta_0(w)$ turns (5), (6) to Lyapunov inequalities.

Summary: convex analysis and convex and saddle conjugacy ...

are useful in control systems theory beyond optimal control

How about variational analysis?

Variational and Convex Analysis Techniques for Problems Involving Dynamics
International Symposium on Mathematical Programming, Chicago 2009

Variational Analysis in Dynamics and Control
SIAM Conference on Control and Applications, San Diego 2013

Variational Analysis in Dynamics and Control
IEEE Conference on Decision and Control, Los Angeles 2014

Variational Analysis in Dynamics and Control
IEEE Conference on Decision and Control, **Las Vegas 2016**