On the convexity of piecewise-defined functions

Yves Lucet
Heinz H. Bauschke (UBC) and Hung M. Phan (UMass Lowell)
Supported by NSERC DG/Accelerator & UMass Lowell

Dedicated to Terry on the occasion of his 80th birthday

Outline

Convex Transforms

$$f^*(s) = \sup_x \langle s, x\rangle - f(x)$$

$$M_\lambda f(x) = \inf_y [f(y) + \frac{\|x-y\|^2}{2\lambda}]$$

$$h_{\mu,\lambda} f(x) = -M_\mu(-M_\lambda f(x))$$

$$\mathcal{P}_\lambda(f_0, f_1) = [(1-\lambda)M_1(f_0^*) + \lambda M_1(f_1^*)]^* - \frac{1}{2}\|\cdot\|^2$$

$$p_\mu(f_0, f_1; \lambda) = -M_{\mu+\lambda(1-\lambda)}\left(-[(1-\lambda)M_\mu f_0 + \lambda M_\mu f_1]\right)$$

$$k_\lambda(f_1, f_2)(x) = \inf_{(1-\lambda_0)y_0+\lambda y_1 = x} [(1-\lambda)f_0 + \lambda f_1 + \lambda(1-\lambda)g(y_0 - y_1)]$$

Convex Operators

Core

- Addition, scalar multiplication
- Fenchel Conjugate or Moreau envelope
- Convex Envelope

Composite

- Lasry-Lions double envelope
- Proximal Average

History

Fast Algorithms

FLT Brenier 89, Corrias 96, Lucet 96, Noullez 94, She 92, Deniau 95

LLT Lucet 97

PE Felzenszwalb 04

NEP Lucet 06, Moreau 65

PLT Hiriart-Urruty 07

PLQ

PLQ Lucet 06

GPH Gardiner 11, Goebel 08

CO Gardiner 10

Fit Gardiner 09

Piecewise Linear-Quadratic Functions

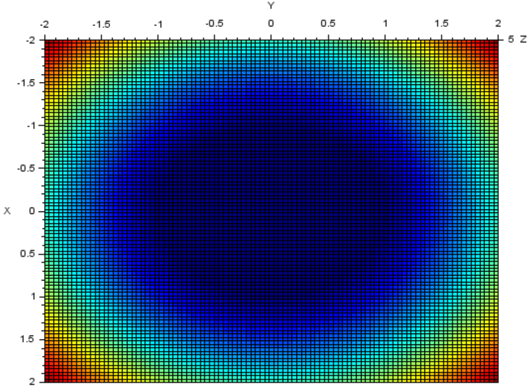
Definition

- Domain is the intersection of linear functions
- On each piece, the function is quadratic
- Restrict to continuous functions on $\operatorname{ri} \operatorname{dom} f$.

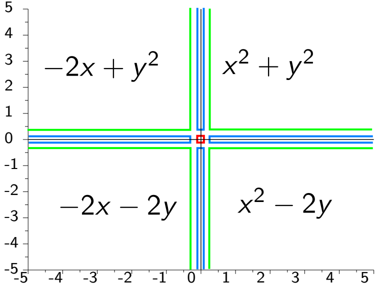
Properties

- $\surd$ Closed class under convex operators
- $\surd$ Hybrid symbolic numerical algorithms running in Linear-time.
- $\surd$ Even nicer for univariate functions.
- **X** Convex envelope of a PLQ function may not be PLQ
- **X** Maximum of two convex PLQ functions may not be PLQ

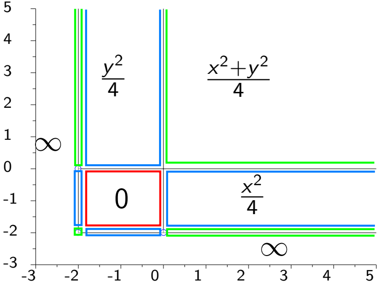
Computational Convex Analysis



Conjugate Entities

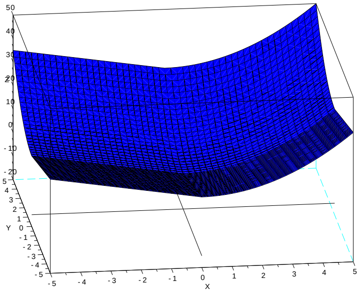


A full-domain model.

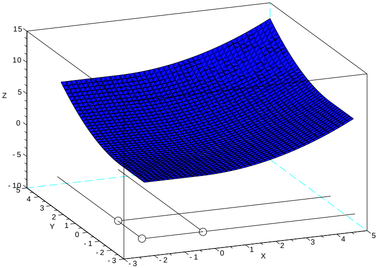


The conjugate model.

Conjugate Entities



A full-domain model.



The conjugate model.

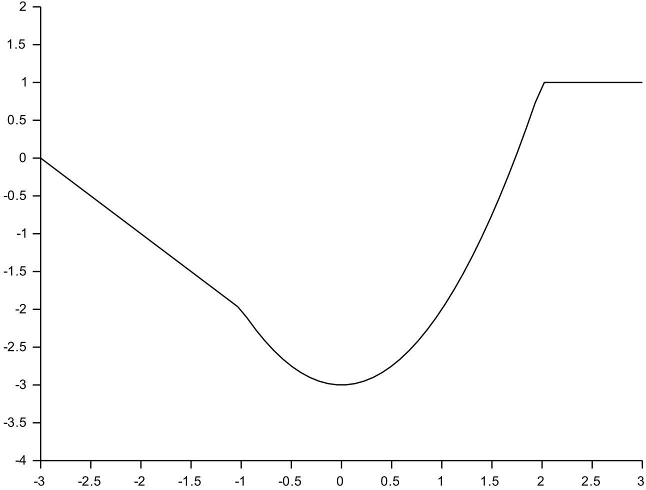
$f : \mathbb{R}^2 \to \mathbb{R}$ convex?

$$f(x,y) := \begin{cases} \frac{x^2+y^2+2\max\{0,xy\}}{|x|+|y|}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{otherwise.} \end{cases}$$

f component-wise convex

- $f_1(x,y) := x + y$ on $A_1 := \mathbb{R}_+ \times \mathbb{R}_+$
- $f_2(x,y) := \frac{x^2+y^2}{-x+y}$ on $A_2 := \mathbb{R}_- \times \mathbb{R}_+$
- $f_3(x,y) := -x - y$ on $A_3 := \mathbb{R}_- \times \mathbb{R}_-$
- $f_4(x,y) := \frac{x^2+y^2}{x-y}$ on $A_4 := \mathbb{R}_+ \times \mathbb{R}_-$

Example



Outline

Lemma

Let $f : X \to]-\infty, +\infty]$ *and let* z_1 *and* z_2 *be in* D_f*. Set* $x := (1-t)z_1 + tz_2$*, where* $t \in [0,1]$*, and assume that* $\partial f(x) \neq \emptyset$*. Then* $f(x) \leq (1-t)f(z_1) + tf(z_2)$*.*

Proof

Let $x^* \in \partial f(x)$.

$$\begin{aligned}(1-t)\langle x^*, z_1 - x\rangle &\leq (1-t)(f(z_1) - f(x))\\ t\langle x^*, z_2 - x\rangle &\leq t(f(z_2) - f(x))\\ 0 &\leq (1-t)f(z_1) + tf(z_2) - f(x)\end{aligned}$$

Zălinescu 2002, Theorem 2.4.1(iii)

If $\mathrm{D}_f = \mathrm{D}_{\partial f}$ is convex then f is convex.

Add continuity, remove finite set

Lemma

Let $f\colon X \to \,]-\infty, +\infty]$ *be proper. Assume*

- $f\big|_{\mathrm{D}_f}$ *is continuous*
- D_f *is convex and at least 2-dimensional*
- *there exists a finite subset* E *of* X *such that* $f\big|_{[x,y]}$ *is convex for every segment* $[x, y]$ *contained in* $(\operatorname{ri} \mathrm{D}_f) \smallsetminus E$

Then f *is convex.*

Fails in dimension 1

$f(x) = -|x|$ and $E = \{0\}$ not dim 2 and not convex

compatible systems of sets

Assume I is a nonempty finite set and $\mathcal{A} := \{A_i\}_{i\in I}$ is a system of convex subsets of X. Note $A := \bigcup_{i\in I} A_i$

$\mathcal{A}$ is a compatible system of sets if

$$\left.\begin{array}{r} i\in I \\ j\in I \\ i\neq j \end{array}\right\} \quad\Rightarrow\quad \operatorname{cl} A_i \cap \operatorname{cl} A_j \cap \operatorname{ri} A = A_i \cap A_j \cap \operatorname{ri} A$$

Examples

- Any finite system of closed convex sets is compatible
- $A_1 =]0,1] \times [0,1]$, $A_2 = [-1,0] \times [0,1]$. Then $A = A_1 \cup A_2 = [-1,1] \times [0,1]$ and $\operatorname{ri} A =]-1,1[\times]0,1[$. Thus, $\mathcal{A} = \{A_1, A_2\}$ is incompatible because

$$\operatorname{cl} A_1 \cap \operatorname{cl} A_2 \cap \operatorname{ri} A = \{0\} \times]0,1[\neq \varnothing = A_1 \cap A_2 \cap \operatorname{ri} A.$$

Colinearly ordered tuple

colinearly ordered tuple

The tuple of vectors $(x_0, \ldots, x_n) \in X^n$ is said to be colinearly ordered if

1. $[x_0, x_n] = [x_0, x_1] \cup \cdots \cup [x_{n-1}, x_n]$
2. $0 \leq \|x_0 - x_1\| \leq \|x_0 - x_2\| \leq \cdots \leq \|x_0 - x_n\|$.

Proposition

Assume $\mathcal{A}$ is a compatible system of sets Then for every segment $[x, y]$ contained in $\operatorname{ri} A$, there exists a colinearly ordered tuple $(x_0, \ldots, x_n)$ and $\{A_{i_1}, \ldots, A_{i_n}\} \subseteq \mathcal{A}$ such that

$$x_0 = x; \quad x_n = y; \quad \text{and} \quad \big(\forall k \in \{1, \ldots, n\}\big) \quad [x_{k-1}, x_k] \subseteq A_{i_k}.$$

Compatible System of Functions

Assume

- I is a nonempty finite
- $\mathcal{F} := \{f_i\}_{i\in I}$ is a system of proper convex functions
- $f := \min_{i\in I} f_i$ is piecewise-defined associated with $\mathcal{F}$
- $I_{\mathcal{F}}(x) = \left\{i \in I \;\middle|\; x \in \mathrm{D}_{f_i}\right\}$ is the active index set function.

$\mathcal{F}$ compatible system of functions if $f_i\big|_{D_{f_i}}$ is continuous and

$$\mathrm{D}_{f_i} \cap \mathrm{D}_{f_j} \neq \varnothing \quad \Rightarrow \quad f_i\big|_{\mathrm{D}_{f_i}\cap\mathrm{D}_{f_j}} \equiv f_j\big|_{\mathrm{D}_{f_i}\cap\mathrm{D}_{f_j}}.$$

Outline

Compatible System

Assume $\mathcal{F}$ is compatible system of functions then

$$\partial f(x) \subseteq \bigcap_{i \in I_{\mathcal{F}}(x)} \partial f_i(x)$$

Colinearly ordered

- $\mathcal{F}$ is compatible system of functions
- (a, b, c) is colinearly ordered, $\mathrm{D}_{f_1} = [a, b]$, $\mathrm{D}_{f_2} = [b, c]$
- $\partial f_1(b) \cap \partial f_2(b) \neq \varnothing$

Then f is convex and

$$\partial f(x) = \bigcap_{i \in I_{\mathcal{F}}(x)} \partial f_i(x) = \begin{cases} \partial f_1(x), & \text{if } x \in [a, b[\,; \\ \partial f_1(b) \cap \partial f_2(b), & \text{if } x = b; \\ \partial f_2(x), & \text{if } x \in\,]b, c]\,. \end{cases}$$

Main Result I

- $\mathcal{F}$ is a compatible system of functions
- $\mathrm{D}_f = \bigcup_{i\in I} \mathrm{D}_{f_i}$ is convex and at least 2-dimensional.
- $\{\mathrm{D}_{f_i}\}_{i\in I}$ is a compatible system of sets
- $\exists E \subset X$ finite
  $x \in (\operatorname{ri} \mathrm{D}_f) \smallsetminus E$ and $|I(x)| \geq 2 \Rightarrow \quad \bigcap_{i\in I(x)} \partial f_i(x) \neq \varnothing$

Then f is convex and

$$(\forall x \in \operatorname{ri} \mathrm{D}_f) \quad \varnothing \neq \partial f(x) \subseteq \bigcap_{i\in I(x)} \partial f_i(x).$$

Ignore Finite number of points

- No need to check subdifferential nonemptyness at a finite number of points
- f convex implies subdifferential nonempty even at these (unchecked) points

Domain compatibility is required

$$f_1 = \iota_{[0,1]\times[0,1]} \quad \text{and} \quad f_2 = \iota_{[-1,0[\times[0,1]} + 1.$$

- $\mathcal{F}$ is a compatible system of functions.
- $\mathrm{D}_f = [-1,1] \times [0,1]$ is convex
- $\{\mathrm{D}_{f_i} \cap \operatorname{ri} \mathrm{D}_f\}_{i\in I}$ is *not* a compatible system of sets.
- f is not convex.

Main Result II

- $\mathcal{F}$ is a compatible system of functions
- f_i is differentiable on $\operatorname{int} \mathrm{D}_{f_i} \neq \varnothing$
- $\mathrm{D}_f = \bigcup_{i \in I} \mathrm{D}_{f_i}$ is convex and at least 2-dimensional.
- $\{\mathrm{D}_{f_i}\}_{i \in I}$ is a compatible system of sets
- There exists a finite subset E of X such that

$$\left.\begin{array}{c} x \in (\operatorname{int} \mathrm{D}_f) \smallsetminus E \\ \{i,j\} \subseteq I(x) \end{array}\right\} \Rightarrow \lim_{\substack{z \to x \\ z \in \operatorname{int} \mathrm{D}_{f_i}}} \nabla f_i(z) = \lim_{\substack{z \to x \\ z \in \operatorname{int} \mathrm{D}_{f_j}}} \nabla f_j(z) \quad \text{exists.}$$

Then f is convex and C^1 on $(\operatorname{int} \mathrm{D}_f) \smallsetminus E$.

Corollary

- $\mathcal{F}$ is a compatible system of functions
- $\mathrm{D}_f = \bigcup_{i \in I} \mathrm{D}_{f_i}$ is convex and at least 2-dimensional.
- $\{\mathrm{D}_{f_i}\}_{i \in I}$ is a compatible system of sets
- f is continuously differentiable on $\operatorname{int} \mathrm{D}_f$.

Then f is convex.

PLQ functions

- $\mathcal{F}$ is a compatible system of functions
- D_{f_i} is closed, $f_i(x) = 1/2\langle x, A_i x\rangle + \langle b_i, x\rangle + \gamma_i$
- $A_i = A_i^T \succeq 0$
- $\mathrm{D}_f = \bigcup_{i\in I} \mathrm{D}_{f_i}$ is convex and at least 2-dimensional.

$$\{i,j\} \subseteq I, i \neq j, x \in \mathrm{D}_{f_i} \cap \mathrm{D}_{f_j} \Rightarrow A_i x + b_i = A_j x + b_j.$$

Then f is convex.

CPLQ Structure

Jie Sun, *On the structure of convex piecewise quadratic functions* JOTA 1992

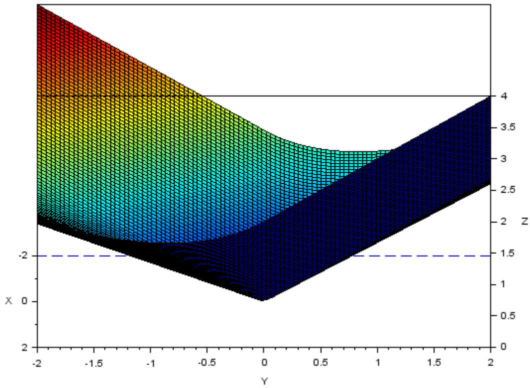
$f : \mathbb{R}^2 \to \mathbb{R}$ convex?

$$f(x,y) := \begin{cases} \frac{x^2+y^2+2\max\{0,xy\}}{|x|+|y|}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{otherwise.} \end{cases}$$

- $\{f_i\}_{i\in I}$ is a compatible system of functions
- $\{D_{f_i}\}_{i\in I} = \{A_i\}_{i\in I}$ is a compatible system of sets
- f_i is differentiable on int A_i
- Hessian of each f_i is positive semi-definite on A_i
- $$\lim_{\substack{(x,y)\to(a,0)\\(x,y)\in \text{int } A_1}} \nabla f_1(x,y) = \lim_{\substack{(x,y)\to(a,0)\\(x,y)\in \text{int } A_4}} \nabla f_4(x,y) = (1,1)$$

Take $E := \{(0,0)\}$ we deduce f convex, C^1 away from $(0,0)$.

Example



Outline

Conclusion

- Sufficient condition for convexity
- Only needs to check boundaries
- Can ignore finite number of points

References

- H. H. Bauschke, Y. Lucet, H. M. Phan. On the convexity of piecewise-defined functions. Accepted in ESAIM: Control, Optimisation and Calculus of Variations
- Y. Lucet, *Techniques and Open Questions in CCA* 2013
- Y. Lucet, *What Shape is your Conjugate?* SIREV 2010
- J. Sun, *On the structure of convex piecewise quadratic functions* JOTA 1992

http://atoms.scilab.org/toolboxes/CCA

Computational Convex Analysis

(22/2803 downloads)

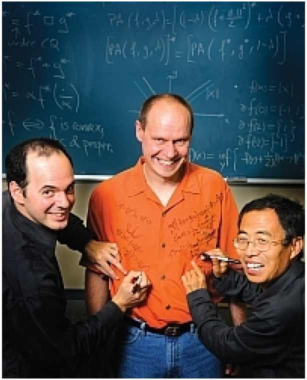
Algorithms for manipulating and visualizing convex functions.

Details

Version	1.6.2-2
Author(s)	Yves Lucet
Entity	University of British Columbia - Okanagan
Package maintainer	Yves Lucet
Categories	Optimization
	Modeling and Control Tools
	Optimization - General
WebSite	https://people.ok.ubc.ca/ylucet/cca.php
License	GPL (3.0)
Supported Scilab Versions	>= 5.4
Creation Date	10th of May 2015
ATOMS packaging system	Available on
How To Install	atomsInstall('CCA')

West Coast Optimization Meeting

- Kelowna, BC, Canada
- October 10, 2015
- https://ocana.ok.ubc.ca/seminars.php



Happy Birthday Terry