A SURVEY OF RESULTS ON LINEAR CONVERGENCE FOR ITERATIVE PROXIMAL ALGORITHMS IN NONCONVEX SETTINGS

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Outline

Happy 80th year Terry!

Goals

Solve

$$0 \in F(x)$$

for $F : \mathbb{E} \rightrightarrows \mathbb{E}$ with $\mathbb{E}$ a Euclidean space.

- #1. Convergence (with rates and radii) of Picard iterations:

$$x^{k+1} \in Tx^k$$

- #2. Algorithms:
  - Feasibility: alternating projections
  - Optimization: Douglas-Rachford
- #2. Applications:
  - Ptychography/Phase retrieval
  - Rank constrained affine feasibility
  - Optimization with high-dimensional structured constraints

Building blocks

- Resolvent: $(\mathrm{Id} + \lambda F)^{-1}$
- Prox operator: for a function $f : X \to \overline{\mathbb{R}}$, define

$$\mathrm{prox}_{\eta f}(x) := \mathrm{argmin}_{\,y}\, f(y) + \frac{1}{2\eta}\|y - x\|^2$$

- Proximal reflector: $R_{\eta f} := 2\,\mathrm{prox}_{\eta f} - \mathrm{Id}$
- Projector: if $f = \iota_\Omega$ for $\Omega \subset X$ closed and nonempty, then $\mathrm{prox}_{\eta f}(\overline{x}) = P_\Omega \overline{x}$ where

$$\begin{aligned} P_\Omega x &:= \{\overline{x} \in \Omega \mid \|x - \overline{x}\| = \mathrm{dist}\,(x, \Omega)\} \\ \mathrm{dist}\,(x, \Omega) &:= \inf_{y \in \Omega} \|x - y\|. \end{aligned}$$

- Reflector: if $f = \iota_\Omega$ for some closed, nonempty set $\Omega \subset X$, then $R_\Omega := 2P_\Omega - \mathrm{Id}$

Algorithms

- Proximal point: $T_{PP} := (\mathrm{Id} + \lambda F)^{-1}$ (fixed step length)

and for $F = \partial f_1 + \partial f_2$, splitting algorithms:

- Forward-backward/Projected (sub)gradients:
  $T_{FB} := \mathrm{prox}_{f_1}(\mathrm{Id} - \lambda \partial f_2)$
- Backward-backward/Alternating projections: $T_{AP} := \mathrm{prox}_{f_1}\mathrm{prox}_{f_2}$
- Douglas-Rachford (ADMM):
  $T_{DR} := \mathrm{prox}_{f_1}\left(2\,\mathrm{prox}_{f_2} - \mathrm{Id}\right) - \mathrm{prox}_{f_2} + \mathrm{Id}$

Convergence analysis

Monotone/Convex

- Monotone proximal point: Martinet, Rokafellar
- Convex forward-backward: Eckstein, Combettes-Wajs, Combettes-Pesquet, Combettes-Dung-Vu, Nesterov, Beck-Teboulle
- Convex alternating projections: von Neumann, Aronszajn, Cheney-Goldstein, Bregman, Gubin-Polyak-Raik
- Convex Douglas-Rachford/ADMM: Lions-Mercier, Gabay

Nonmonotone/Nonconvex

- Nonmonotone proximal point: Spingarn, Pennanen, Combettes-Pennanen, Iusem-Pennanen-Svaiter, Aragón-Geoffroy, Aragón-Dontchev-Geoffroy, Attouch-Bolte
- Nonconvex alternating projections: Combettes-Trussell, Lewis-Malick, Lewis-L.-Malick, Bauschke-L.-Phan-Wang, **Hesse-L.**, Hesse-L.-Neumann, **Noll-Rondepierre**
- Nonconvex forward-backward: Attouch-Bolte-Redont-Soubeyran
- Nonconvex Douglas-Rachford: Borwein-Sims, **Hesse-L.**, Hesse-L.Neumann, Phan, Bauschke-Noll

Outline

Ambient regularity

(S, ϵ)-nonexpansive mappings

Let D and S be nonempty subsets of $\mathbb{E}$ and let T be a (multi-valued) mapping from D to $\mathbb{E}$.

- T is called (S, ε)-nonexpansive on D if, at each $\overline{x} \in S$, for all $x \in D$

$$\|x_+ - \overline{x}_+\| \leq \sqrt{1 + \varepsilon}\, \|x - \overline{x}\|$$
$$\forall\, \overline{x}_+ \in T\overline{x} \text{ and } \forall\, x_+ \in Tx.$$

- If this holds with $\epsilon = 0$ then T is called S-nonexpansive on D.
- If this holds with $\epsilon = 0$ and $S = D$ then T is called nonexpansive on D.

(S, ϵ)-firmly nonexpansive mappings

Let D and S be nonempty subsets of $\mathbb{E}$ and let T be a (multi-valued) mapping from D to $\mathbb{E}$.

- T is called (S, ε)-firmly nonexpansive on D if, at each $\overline{x} \in S$, for all $x \in D$

$$\|x_+ - \overline{x}_+\|^2 + \|(x - x_+) - (\overline{x} - \overline{x}_+)\|^2 \leq (1 + \varepsilon) \|x - \overline{x}\|^2$$
$$\forall\, \overline{x}_+ \in T\overline{x} \text{ and } \forall x_+ \in Tx.$$

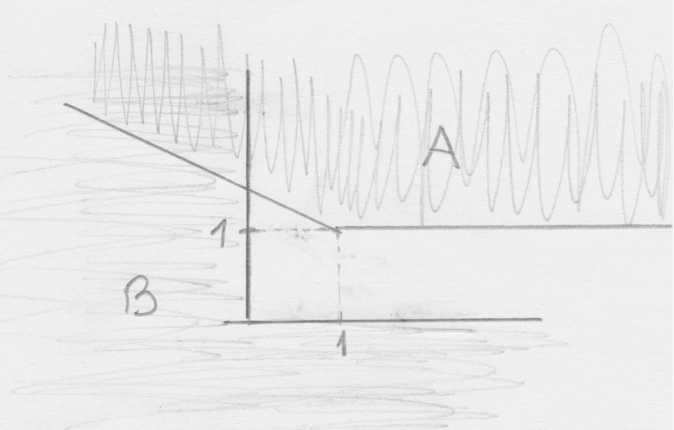
- If this holds with $\epsilon = 0$ then T is called S-firmly nonexpansive on D.
- If this holds with $\epsilon = 0$ and $S = D$ then T is called firmly nonexpansive on D.

Fixed points of set-valued mappings

The set of fixed points of a set-valued mapping $T : X \rightrightarrows X$ is defined by

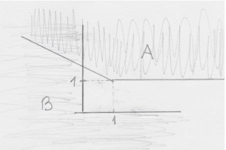
$$\{x \in X \mid x \in T(x)\}.$$

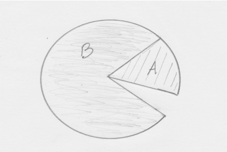
When $T := P_A P_B$ this includes the point $\{(1,1)\}$ in



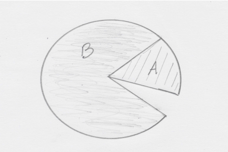
Alternating Projections: $T = P_A P_B$

(i) IS NOT (Fix T, ϵ)-firmly nonexpansive on A for any ϵ

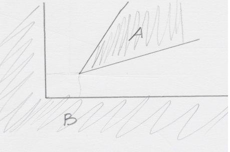




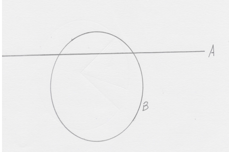
(ii) IS (S, ϵ)-firmly nonexpansive on A for



$(S, \epsilon) = (\{(0,0)\}, 0)$



$(S, \epsilon) = (\{(1,1)\}, 0)$



$(S, \epsilon) = (\{a^*\}, 1)$

$\square$

Regularity at the fixed point set

Basic Lemma

Let $T : D \rightrightarrows \mathbb{E}$ for $D \subset \mathbb{E}$ and let $S \subset \operatorname{Fix} T \subset \operatorname{ri} D$. Define $S_\delta := \delta\mathbb{B} + S \subset D$ for $\delta \in [0, \overline{\delta})$ fixed. Suppose

(a) T is (S, ϵ)-firmly nonexpansive on $S_{\overline{\delta}}$, and

(b) there exists $\lambda > 0$ such that T satisfies

$$\|x - x_+\| \geq \lambda \operatorname{dist}(x, S) \quad \forall\, x_+ \in Tx,\ \forall\, x \in S_{\overline{\delta}} \setminus \left(S_{\overline{\delta}} \cap (\operatorname{Fix} T + \delta\mathbb{B})\right). \tag{1}$$

Then, for all $x \in S_{\overline{\delta}} \setminus \left(S_{\overline{\delta}} \cap (\operatorname{Fix} T + \delta\mathbb{B})\right)$

$$\operatorname{dist}(x_+, \operatorname{Fix} T) \leq \sqrt{1 + \epsilon - \lambda^2}\, \operatorname{dist}(x, S) \quad \forall\, x_+ \in Tx. \tag{2}$$

(sub) Linear Convergence

Let $T : D \rightrightarrows D$ for $D \subset \mathbb{E}$ and let $S \subset \mathrm{Fix}\ T \subset \mathrm{ri}\, D$ be closed and nonempty. Define $S_\delta := (\delta\mathbb{B} + S) \cap D$. Suppose that for $\gamma \in [0, 1)$ fixed and for all $\overline{\delta} > 0$ small enough, there is a triplet $(\epsilon, \delta, \lambda) \in \mathbb{R}_+ \times [0, \gamma\overline{\delta}] \times (\sqrt{\epsilon}, \sqrt{\epsilon + 1}]$ such that

(a) T is (S, ϵ)-firmly nonexpansive on $S_{\overline{\delta}}$ and

(b) $\|x - x_+\| \geq \lambda\, \mathrm{dist}(x, S) \quad \forall x_+ \in Tx,\ \forall x \in S_{\overline{\delta}} \setminus \left(S_{\overline{\delta}} \cap (\mathrm{Fix}\ T + \delta\mathbb{B})\right)$.

Then for any x^0 close enough to S the iterates $x^{k+1} \in T(x^k)$ satisfy $\mathrm{dist}(x^k, \mathrm{Fix}\ T) \to 0$ as $k \to \infty$.

Condition (a) is relatively easy to verify. What about (b)?

The key insight into condition (b) is the connection to metric SUBregularity of set-valued mappings. This approach to the study of algorithms has been advanced by several authors [Pennanen, Klatte&Kummer, Aragón&Geoffroy, Dontchev&Rockafellar]

(strong) Metric (sub)-Regularity

(i) The mapping $\Phi : \mathbb{E} \rightrightarrows \mathbb{Y}$ is called metrically regular at of order w $\overline{x}$ for $\overline{y}$ if there is a constant $\kappa > 0$ together with neighborhood U of $\overline{x}$ and V of $\overline{y}$ such that

$$\operatorname{dist}\left(x, \Phi^{-1}(y)\right) \leq \kappa \operatorname{dist}^{w}\left(y, \Phi(x)\right) \quad \forall (x, y) \in U \times V. \tag{3}$$

The regularity modulus is the infimum of those constants $\kappa > 0$ such that (3) holds.

(ii) The mapping $\Phi : \mathbb{E} \rightrightarrows \mathbb{Y}$ is called metrically subregular of order w at $\overline{x}$ for $\overline{y}$ if $(\overline{x}, \overline{y}) \in \operatorname{gph} \Phi$ and there is a constant $\kappa > 0$ and neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that

$$\operatorname{dist}\left(x, \Phi^{-1}(\overline{y})\right) \leq \kappa \operatorname{dist}^{w}\left(\overline{y}, \Phi(x) \cap V\right) \quad \forall\, x \in U. \tag{4}$$

The subregularity modulus is the infimum of those constants $\kappa > 0$ such that (4) holds.

(sub) Linear Convergence with Metric Subregularity

Let $T : D \rightrightarrows D$ for $D \subset \mathbb{E}$, $\Phi := T - \mathrm{Id}$ and let $S \subset \mathrm{Fix}\ T \subset \mathrm{ri}\, D$ be closed and nonempty. Suppose that, for any $\overline{\delta} > 0$ small enough, there are $\gamma \in (0,1)$, $w \in (0,1]$, a nonnegative sequence of scalars $(\epsilon_i)_{i\in\mathbb{N}}$ and a constant $\kappa > 0$, such that, for all $i \in \mathbb{N}$, $S_{\overline{\delta}} := \left(\overline{\delta}\mathbb{B} + S\right) \cap D$,

$$\frac{1}{\sqrt{\epsilon_i + 1}} \leq \kappa \inf_{x \in S_{\gamma^i \overline{\delta}} \setminus \left(\left(\mathrm{Fix}\ T + \gamma^{i+1}\overline{\delta}\mathbb{B}\right) \cap S_{\gamma^i \overline{\delta}}\right)} \left\{ \inf_{x_+ \in Tx} \|x_+ - x\|^{w-1} \right\} < \frac{1}{\sqrt{\epsilon_i}}, \tag{5}$$

and

(a) T is (S, ϵ_i)-firmly nonexpansive on $S_{\gamma^i \overline{\delta}}$ and

(b) $\mathrm{dist}\left(x, \Phi^{-1}(0)\right) \leq \kappa\, \mathrm{dist}^w\left(0, \Phi(x)\right) \quad \forall x \in S_{\gamma^i \overline{\delta}} \setminus \left(\left(\mathrm{Fix}\ T + \gamma^{i+1}\overline{\delta}\mathbb{B}\right) \cap S_{\gamma^i \overline{\delta}}\right)$.

(sub) Linear Convergence with Metric Subregularity

Then, for any $x^0 \in S_{\overline{\delta}}$, the iterates $x^{j+1} \in T(x^j)$ converge to $\mathrm{Fix}\ T$ with

$$\mathrm{dist}\left(x^{j+1}, \mathrm{Fix}\ T\right) \leq \sqrt{1 + \epsilon_i - \left(\frac{1}{\kappa_i}\right)^2}\,\mathrm{dist}\left(x^j, S\right) \tag{6}$$

for all $x^j \in S_{\gamma^i \overline{\delta}} \setminus \left(\left(\mathrm{Fix}\ T + \gamma^{i+1}\overline{\delta}\mathbb{B}\right) \cap S_{\gamma^i \overline{\delta}}\right)$ where

$$\kappa_i := \kappa \inf_{x \in S_{\gamma^i \overline{\delta}} \setminus \left(\left(\mathrm{Fix}\ T + \gamma^{i+1}\overline{\delta}\mathbb{B}\right) \cap S_{\gamma^i \overline{\delta}}\right)} \left\{ \inf_{x_+ \in Tx} \|x_+ - x\|^{w-1} \right\}. \tag{7}$$

In particular, if $w = 1$, then convergence is at least linear with rate bounded above by $\sqrt{1 + \overline{\epsilon} - \left(\frac{1}{\kappa}\right)^2}$ where $\overline{\epsilon} := \sup_k \{\epsilon_k\}$.

Outline

Projection algorithms for feasibility

Regularity of sets

(i) A nonempty set $\Omega \subset \mathbb{E}$ is *(ε, δ)-subregular* at $\widehat{x}$ with respect to $S \subset \mathbb{E}$ if

$$\langle v, \overline{x} - y \rangle \leq \varepsilon \, \|v\| \, \|\overline{x} - y\| \tag{8}$$

holds for all $y \in \mathbb{B}_\delta(\widehat{x}) \cap \Omega$, $\overline{x} \in S \cap \mathbb{B}_\delta(\widehat{x})$, $v \in N^{\mathrm{prox}}_\Omega(y)$. The set Ω is said to be *(ε, δ)-subregular at $\widehat{x}$* if $S = \{\widehat{x}\}$.

(ii) If $S = \Omega$ in (i) then the set Ω is said to be *(ε, δ)-regular* at $\widehat{x}$.

(iii) A nonempty (locally) closed set $\Omega \subset \mathbb{E}$ is *Clarke regular* at a point $\overline{x} \in \Omega$ if, for all $\epsilon > 0$, $\exists\ \delta > 0$ such that $z \in \mathbb{B}_\delta(\widehat{x})$ and $x \in \Omega \cap \mathbb{B}_\delta(\widehat{x})$, and any $y \in P_\Omega(z)$,

$$\left\langle x - \widehat{x}, z - y \right\rangle \leq \epsilon \|x - \widehat{x}\| \|z - y\|.$$

(iv) If, for all $\epsilon > 0$, there exists a $\delta > 0$ such that (8) holds for all $y, \overline{x} \in \mathbb{B}_\delta(\widehat{x}) \cap \Omega$ and $v \in N^{\mathrm{prox}}_\Omega(y)$, then Ω is said to be *super-regular*.

(v) A nonempty (locally) closed set $\Omega \subset \mathbb{E}$ is *prox-regular* at a point $\overline{x} \in \Omega$ if the projector P_Ω is single-valued in a vicinity of $\overline{x}$.

Set regularity relations

(i) $\{(\epsilon, \delta) - \text{subregular sets}\} \supset \{(\epsilon, \delta) - \text{regular sets}\}$

(ii) $\{(\epsilon, \delta) - \text{regular sets}\} \supset \{\text{Clarke regular sets}\}$

(iii) $\{\text{Clarke regular sets}\} \supset \{\text{super-regular sets}\}$

(iv) $\{\text{super-regular sets}\} \supset \{\text{prox-regular sets}\}$

(v) $\{\text{prox-regular sets}\} \supset \{\text{closed convex sets}\}$

A more recent definition of 0-Hölder regular sets (relative to some other set) developed by Noll&Rondepierre (2015) satisfies

$$\{\text{0-Hölder regular sets}\} \supset \{(\epsilon, \delta) - \text{regular sets}\}$$

but haven't worked out the relation to $\{(\epsilon, \delta)\}$-subregular sets yet.

Projectors and reflectors onto (ε, δ)-subregular sets

Let $\Omega \subset \mathbb{E}$ be nonempty closed and (ε, δ)-subregular at $\widehat{x}$ with respect to $S \subseteq \Omega \cap \mathbb{B}_\delta(\widehat{x})$ and define

$$U := \left\{ x \in \mathbb{E} \;\middle|\; P_\Omega x \subset \mathbb{B}_\delta(\widehat{x}) \right\}. \tag{9}$$

- **(i)** The projector P_Ω is $(S, \widetilde{\varepsilon}_1)$-nonexpansive on U where $\widetilde{\varepsilon}_1 := 2\varepsilon + \varepsilon^2$.
- **(ii)** The projector P_Ω is $(S, \widetilde{\varepsilon}_2)$-firmly nonexpansive on $\mathbb{B}_\delta(\widehat{x})$ where $\widetilde{\varepsilon}_2 := 2\varepsilon + 2\varepsilon^2$.
- **(iii)** The reflector R_Ω is $(S, \widetilde{\varepsilon}_3)$-nonexpansive on $\mathbb{B}_\delta(\widehat{x})$ where $\widetilde{\varepsilon}_3 := 4\varepsilon + 4\varepsilon^2$.

Regularity of collections of sets

Linearly focusing collections of sets

The collection $\{\Omega_1, \Omega_2\}$ of closed subsets of $\mathbb{E}$ is said to be *Ω_1-weakly linearly focusing* at $\overline{x}$ when $P_{\Omega_1} P_{\Omega_2} \overline{x} \ni \overline{x}$ and there is a neighborhood U of $\overline{x}$ and a constant κ such that, for all $x \in U$,

$$\operatorname{dist}(x, E_1) \leq \kappa \operatorname{dist}(x, P_{\Omega_1} P_{\Omega_2} x) \tag{10}$$

where $E_1 := \{e \in \Omega_1 \mid P_{\Omega_1} P_{\Omega_2} e \ni e\}$. The collection is said to be *weakly linearly focusing* (no mention of Ω_1 or Ω_2) at $(\overline{x}_1, \overline{x}_2)$ when $(P_{\Omega_1} P_{\Omega_2} \overline{x}_1, P_{\Omega_2} P_{\Omega_1} \overline{x}_2) \ni (\overline{x}_1, \overline{x}_2)$ and there is a neighborhood U of $(\overline{x}_1, \overline{x}_2)$ and a constant κ such that, for all $(x_1, x_2) \in U$,

$$\operatorname{dist}((x_1, x_2), (E_1, E_2)) \leq \kappa \left\| \operatorname{dist}(x_1, P_{\Omega_1} P_{\Omega_2} x), \operatorname{dist}(x_2, P_{\Omega_2} P_{\Omega_1} x) \right\|. \tag{11}$$

where $E_2 := \{f \in \Omega_2 \mid P_{\Omega_2} P_{\Omega_1} e \ni e\}$.
The collection is said to be *strongly linearly focusing* at $\overline{x}$ when $\overline{x} \in \Omega_1 \cap \Omega_2$ and there is a neighborhood U of $(\overline{x}, \overline{x})$ and a constant κ such that, for all $x \in U$ (11) holds. The infimum of all κ such that (11) holds is called the *(weak/strong) focusing modulus*.

$\{\Omega_1, \Omega_2\}$ intersect transversally

$$\Longrightarrow$$

$\{\Omega_1, \Omega_2\}$ is linearly regular at $\bar{x} \in \Omega_1 \cap \Omega_2$

$$\Longrightarrow$$

$\{\Omega_1, \Omega_2\}$ is locally linearly regular at $\bar{x} \in \Omega_1 \cap \Omega_2$

$$\Longleftrightarrow$$

$\{\Omega_1, \Omega_2\}$ is *strongly linearly focusing* at $\bar{x}$

$$\Longleftrightarrow$$

$\psi := \mathbf{T} - \mathrm{Id}$ is metrically subregular at $(\bar{x}, \bar{x}, 0, 0) \in \operatorname{gph} \psi\,(\bar{x}, \bar{x})$

$$\Longrightarrow$$

$\Phi := T - \mathrm{Id}$ is metrically subregular at $(\bar{x}, 0) \in \operatorname{gph} \Phi\,(\bar{x}, \bar{x})$

$$\Longleftrightarrow$$

$\{\Omega_1, \Omega_2\}$ is Ω_1*-weakly linearly focusing* at $\bar{x}$

where $\mathbf{T} := (P_{\Omega_1} P_{\Omega_2}, P_{\Omega_2} P_{\Omega_1})$ and $T := P_{\Omega_1} P_{\Omega_2}$.

(sub) Linear Convergence of Alternating Projections

Let $T_{AP} : A \rightrightarrows A := P_A P_B$ for A, B closed, $\Phi_{AP} := T_{AP} - \mathrm{Id}$ and let $\overline{x} \in E := \{e \in A \mid P_A P_B e \ni e\}$. Suppose that, for any $\overline{\delta} > 0$ small enough, there are $\gamma \in (0, 1)$ and a pair $(\epsilon, \kappa) > 0$ such that

$$\frac{1}{\sqrt{\epsilon + 1}} \leq \kappa < \frac{1}{\sqrt{\epsilon}}, \tag{12}$$

and

(a) T is $(\{\overline{x}\}, \epsilon)$-firmly nonexpansive on $\overline{\delta}\mathbb{B}(\overline{x}) \cap A$ and

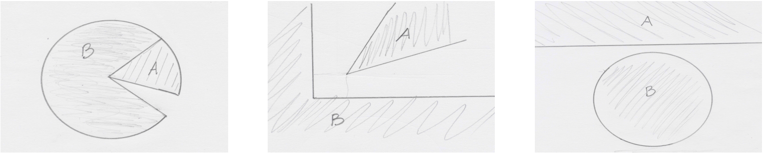
(b) (A, B) is A-weakly linearly focusing at $\overline{x}$.

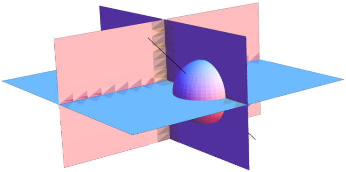
Then, for any $x^0 \in \overline{\delta}\mathbb{B}(\overline{x}) \cap A$, the iterates $x^{j+1} \in Tx^j$ converge linearly to E with rate $\sqrt{1 + \epsilon - \left(\frac{1}{\kappa}\right)^2}$.

Alternating Projections: $T_{AP} := P_A P_B$

Alternating projections converges locally linearly to Fix T_{AP}, with quantifiable rates and regions, in the following instances:

(i) any two lines in $\mathbb{R}^n$ whether they intersect or not.

(ii) 



$\square$

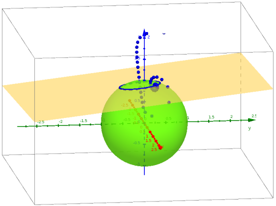
Optimization

Douglas-Rachford: $T_{DR} := \mathrm{prox}_{f_1}\left(2\,\mathrm{prox}_{f_2} - \mathrm{Id}\right) - \mathrm{prox}_{f_2} + \mathrm{Id}$

Let $W := \mathrm{aff}\{x^k\}_{k\in\mathbb{N}}$. The Douglas-Rachford algorithm converges locally linearly to $\mathrm{Fix}\ T_{DR} \cap W$ with quantifiable rates and regions, in the following instances:

(i) f_1 and f_2 are the indicator functions of sets in $\mathbb{R}^n$ with super-regular boundaries and the sets intersect essentially transversally: $T_1(\overline{x}) + T_2(\overline{x}) = W$. (For example, any two intersecting lines in $\mathbb{R}^n$.)

(ii) f_1 and f_2 are convex, piecewise linear-quadratic functions and that $\mathrm{Fix}\ T_{DR} \cap W \cap \mathcal{O} = \{\overline{x}\}$, i.e. is an isolated point. The rate of linear convergence is bounded above by $\sqrt{1-\kappa}$, where $\kappa = c^{-2} > 0$, for c a constant of metric subregularity of $T_{DR} - \mathrm{Id}$ at $\overline{x}$ for the neighborhood $\mathcal{O}$.

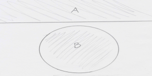


Proof logic.

convex, plc $\Longrightarrow$ single-valued polyhedral prox-operators $\Longrightarrow$ T_{DR} is polyhedral.
T_{DR} polyhedral + Fix $T_{DR} \cap W \cap \mathcal{O}$ an isolated point $\Longrightarrow$ metric subregularity [Dontchev& Rockafellar, Propositions 3I.1 and 3I.2]. $\square$

Outline

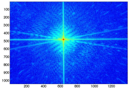
Application: phase retrieval



Phase retrieval

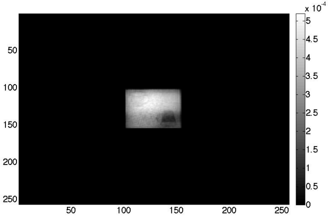
Given

- $|(Ax)_j|^2 = b_j$ for $b_j \in \mathbb{R}_+$ ($j = 1, 2, \ldots, m$) given by

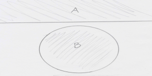


- some qualitative constraint ($x \in {\mathbb{R}^n}_+$ or supp $x \subset D$)

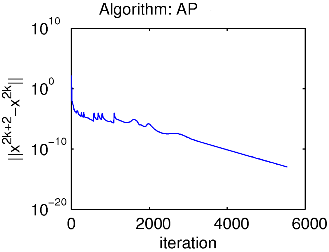
Find

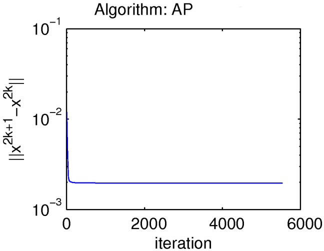


Application: phase retrieval

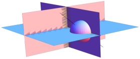


Phase retrieval





Application: sparse affine feasibility [Hesse-L.-Neumann, 2014]



$$\underset{x\in A_s}{\text{minimize}}\ \frac{1}{2}\operatorname{dist}(x,B)^2$$

where $A_s := \left\{x \in \mathbb{R}^n \middle|\ \|x\|_0 \le s\right\}$ and $B := \left\{x \in \mathbb{R}^n \middle|\ Mx = p\right\}$.

- Local convergence of alternating projections and Douglas-Rachford
- Global convergence of alternating projections under the assumption:

$$\begin{gathered} M \text{ is full rank and} \\ (1-\delta_{2s})\,\|x\|_2^2 \le \left\|M^\dagger Mx\right\|_2^2 \quad \forall\, x \in A_{2s} \end{gathered} \tag{13}$$

Denote $A_J := \operatorname{span}\{e_i |\ i \in J\}$ for $J \in \mathcal{J}_{2s} := \left\{J \in 2^{\{1,2,\ldots,n\}} \middle|\ J \text{ has } 2s \text{ elements}\right\}$. Then M satisfies (13) with $\delta_{2s} \in [0, \frac{\alpha-1}{\alpha})$ for some fixed $s > 0$ and $\alpha > 1 \implies$

$$(\forall J \in \mathcal{J}_{2s}) \qquad A_J \cap \ker(M) = \{0\}.$$

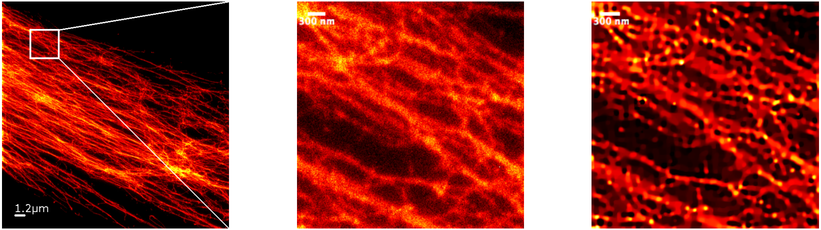
Image denoising/deconvolution

$$\underset{u\in\mathbb{R}^n}{\text{minimize}}\ J(u) + \rho \max\{F_\epsilon(Au)\}.$$

where J is convex, piecewise linear-quadratic, $A:\ \mathbb{R}^n \to \mathbb{R}^n$, and

$$F_\epsilon\ :\ \mathbb{R}^n \to 2^{\mathbb{R}^n}\ := v \mapsto (f_1(v) - \epsilon_1, f_2(v) - \epsilon_2, \ldots, f_M(v) - \epsilon_M)^T \qquad (14)$$

is convex quadratic.



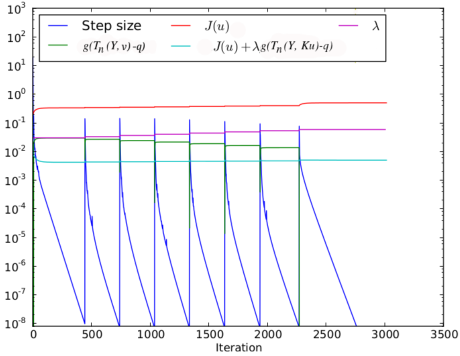
Solve with ADMM = Douglas-Rachford on the dual.

$$\underset{u\in\mathbb{R}^n}{\text{minimize}}\ J(u) + \rho \max\{F_\epsilon(Au)\}.$$

where J is convex, piecewise linear-quadratic, $A : \mathbb{R}^n \to \mathbb{R}^n$, and

$$F_\epsilon : \mathbb{R}^n \to 2^{\mathbb{R}^n} := v \mapsto (f_1(v) - \epsilon_1, f_2(v) - \epsilon_2, \ldots, f_M(v) - \epsilon_M)^T \quad (15)$$

is convex quadratic.



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