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Projections and the Reduction Lemma

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Notation

If P is a nonempty polyhedral convex subset of $\mathbb{R}^n$, then

- Π_P denotes the Euclidean projector on P
- T_P and N_P are respectively the tangent-cone and normal-cone operators of P.
- For subsets X and Y of $\mathbb{R}^n$ the *excess of* X *over* Y is

$$e(X, Y) = \inf\{\alpha \geq 0 \mid X \subset Y + \alpha B^n\},$$

where B^n is the unit ball.

Critical face and critical cone

Definition

Let P be a nonempty polyhedral convex subset of $\mathbb{R}^n$. For any point $z \in \mathbb{R}^n$ let $x = \Pi_P(z)$ and $n^* = z - x$. Then

$$\phi_P(z) := P \cap \{x + [\mathrm{pos}(-n^*)]^\circ\}$$

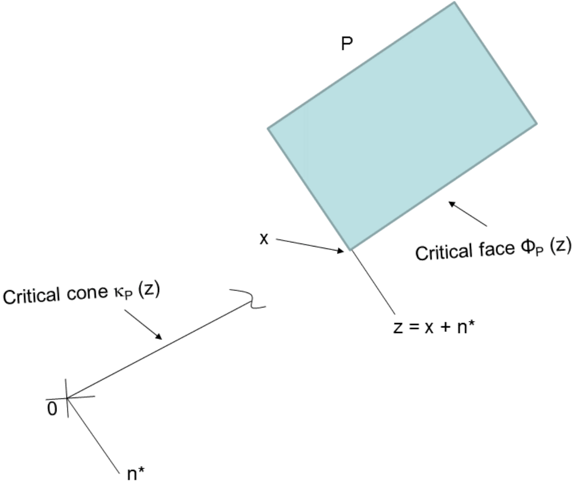
is the *critical face* of P for z. Also,

$$\kappa_P(z) := T_P(x) \cap [\mathrm{pos}(-n^*)]^\circ$$

is the *critical cone* of P for z.

The critical face $\phi_P(z)$ is a nonempty face of P that is also expressible as $N_P^{-1}(n^*)$. The critical cone $\kappa_P(z)$ is a nonempty face of $T_P(x)$.

Illustration



The reduction lemma

Lemma

Let P be a nonempty polyhedral convex subset of $\mathbb{R}^n$ and let $z_0 \in \mathbb{R}^n$. Let $x_0 = \Pi_P(z_0)$ and $n_0^* = z_0 - x_0$. There is a neighborhood Q of the origin in $\mathbb{R}^{2n}$ such that

$$[(x_0, n_0^*) + Q] \cap N_P = (x_0, n_0^*) + [Q \cap N_{\kappa_P(z_0)}]. \qquad (1)$$

This lemma is extremely useful in the local analysis of variational problems, because it lets us change the underlying set from P to $\kappa_P(z_0)$, which often has a simpler structure.

Example

- Variational inequality problem: find $x \in P$ such that $\langle f(q,x), x' - x \rangle \geq 0$ for each $x' \in P$, where q is a parameter and f is continuous.
- Rewrite: $0 \in f(q,x) + N_P(x)$, equivalently $[x, -f(q,x)] \in N_P$.
- Given a solution x_0 for parameter value q_0, suppose we want to investigate parameter values q' near q_0. Write this as $[x, -f(q',x)] \in N_P$.
- The reduction lemma says that for (x,q) close to (x_0,q_0) we can work instead with $[u, -g(q',u)] \in N_V$, where $V = \kappa_P(x_0)$ and $g(q',u) = f(q', x_0 + u) - f(q_0, x_0)$.
- Rewrite the new problem as $0 \in g(q',u) + N_V(x)$; now we have replaced P by the critical cone $V = \kappa_P(x_0)$.

Structure of the proof

- Proof of reduction lemma given in *Math. Oper. Res.* 1991 is very short and simple (13 lines), but it requires a fact about polyhedral convex sets: the sticky face lemma.
- Direct proof in the 2d edition of Dontchev and Rockafellar's book *Implicit Functions and Solution Mappings: A View from Variational Analysis*, does not require the sticky face lemma, but it is longer.
- Unfortunately, the 1984 proof of the sticky face lemma is long, complicated, and not intuitive. We will give a shorter and more accessible proof.
- Having that, we also have a short and accessible proof of the reduction lemma.

The sticky face lemma

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Lemma

Let P be a nonempty polyhedral convex subset of $\mathbb{R}^n$. Let $x_0^* \in \mathbb{R}^n$ and define $F = N_P^{-1}(x_0^*)$. Then there is a neighborhood U of x_0^* such that whenever $x^* \in U$ one has

$$N_P^{-1}(x^*) = N_F^{-1}(x^*). \tag{2}$$

The reason for the lemma's name is that it says the faces $N_P^{-1}(x^*)$ cannot escape from the face F as long as x^* remains near x_0.

This is intuitively clear

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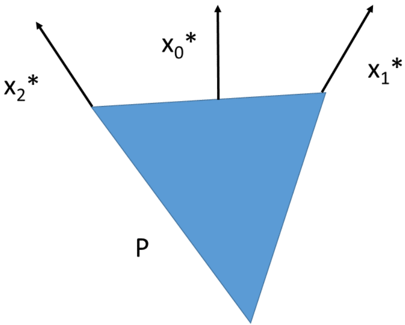

Preparing the proof

We say two subsets X and Y of $\mathbb{R}^n$ are *locally identical* at $z \in \mathbb{R}^n$ if there is a neighborhood N of z such that $X \cap N = Y \cap N$.

A fact about barrier cones:

Lemma

Let P be a convex subset of $\mathbb{R}^n$ such that $\operatorname{bc} P$ *is polyhedral, and let $(x_0, x_0^*) \in N_P$. Let $F = N_P^{-1}(x_0^*)$. Then:*

a. $\operatorname{bc} F$ *is polyhedral: specifically,*

$$\operatorname{bc} F = (\operatorname{bc} P) + \operatorname{pos}\{-x_0^*\}; \tag{3}$$

b. $\operatorname{bc} F$ *and* $\operatorname{bc} P$ *are locally identical at* x_0^*.

The proof of this lemma is straightforward convex analysis, using the properties of barrier cones.

Key arguments

- Dispose of the case $F = \emptyset$
- Use the lemma on the preceding slide to produce a neighborhood Q of x_0^* such that $Q \cap \operatorname{bc} F = Q \cap \operatorname{bc} P$.
- Observe that N_P^{-1} is a *polyhedral multifunction* (i.e., its graph is a finite union of polyhedral convex sets) and therefore there are a nonnegative λ and a neighborhood W of x_0^* such that whenever $x^* \in W$ one has

$$e[N_P^{-1}(x^*), N_P^{-1}(x_0^*)] \leq \lambda \|x^* - x_0^*\|$$

- Observe that among the faces G of P that are not contained in F there is one that minimizes $e(G, F)$ and this minimum ρ is positive

Combining the arguments

- Choose a positive α such that $\lambda\alpha < \rho$ and let

$$U := Q \cap W \cap B(x_0^*, \alpha)$$

- If $x^* \in U$ then $e[N_P^{-1}(x^*), N_P^{-1}(x_0^*)] < \rho$ and therefore $N_P^{-1}(x^*) \subset F$
- If $N_P^{-1}(x^*)$ is empty then so is $N_P^{-1}(x^*)$ $(x^* \in Q)$
- If $N_P^{-1}(x^*)$ is nonempty then $\langle x^*, \cdot \rangle$ has at least one maximizer on P, and all such lie in F and so are also maximizers on F. Then $N_P^{-1}(x^*) = N_F^{-1}(x^*)$ and we are finished

What we've covered

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- The reduction lemma is an important tool for local analysis of variational problems posed over polyhedral convex sets.
- It has a very short proof provided that one has the sticky face lemma.
- We now have a fairly short and geometrically clear proof of the sticky face lemma.