Measures of Residual Risk with Connections to Regression, Risk Tracking, Surrogate Models, and Ambiguity

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Surrogate models: learning from low-fidelity simulations

Output of costly simulation: random variable Y
Output of inexpensive simulation: random variable X

Find f such that $Y \approx f(X)$, or Y "safely" $\leq f(X)$

Surrogate models: learning from low-fidelity simulations

Output of costly simulation: random variable Y
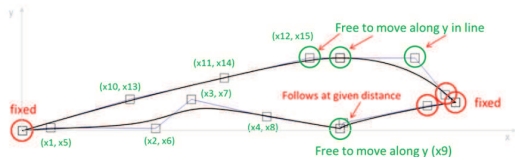Output of inexpensive simulation: random variable X

Find f such that $Y \approx f(X)$, or Y "safely" $\leq f(X)$

Case study: Drag-lift ratio estimation*:

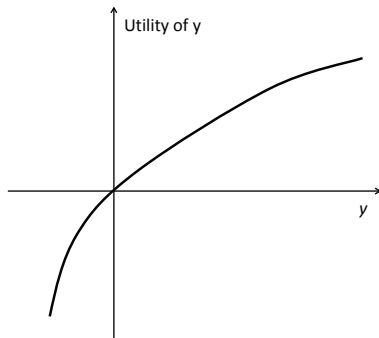Costly: Navier-Stokes solve (4 hours on 8 cores)
Inexpensive: potential flow solve (5 sec on 1 core)

*with S. Brizzolara, Mech. Engineering, MIT
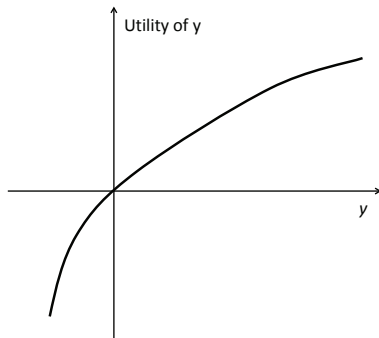
Decision problem: how to invest

Future loss Y
Preferences regarding losses (utility-like functions)

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Utility of y

Opportunities:

Invest in fixed-income asset now

Invest in shares with uncertain value at the future point in time

Balance upfront cost against future (reduced) loss

Outline

- Background: regret, risk, error, deviation
- Measures of residual risk: definition and properties
- Application to surrogate models

Background: regret, risk, error, deviation

Regret

Probability space $(\Omega, \mathcal{F}, P)$;
$\mathcal{L}^2 = \{Y : \Omega \to \mathbf{R} \mid Y \text{ measurable}, E[Y^2] < \infty\}$

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Measure of regret $\mathcal{V} : \mathcal{L}^2 \to (-\infty, \infty]$

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$\mathcal{V}(Y) =$ quantification of displeasure with outcomes of Y

   For example, compensation required for being exposed to Y

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Orientation towards minimization:
For example $\mathcal{V}(Y) = -E[u(-Y)]$ for utility function u

Regularity

$\mathcal{V}$ is regular if: convex

closed

$\mathcal{V}(0) = 0$

$\mathcal{V}(Y) > E[Y]$ when Y not identical to 0

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Measure of risk $\mathcal{R} : \mathcal{L}^2 \to (-\infty, \infty]$

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$$\mathcal{R}(Y) = \text{quantification of the "risk" in } Y$$

$$Y \text{ safely} \leq Y' \iff Y \leq_{\mathcal{R}} Y' \iff \mathcal{R}(Y) \leq \mathcal{R}(Y')$$

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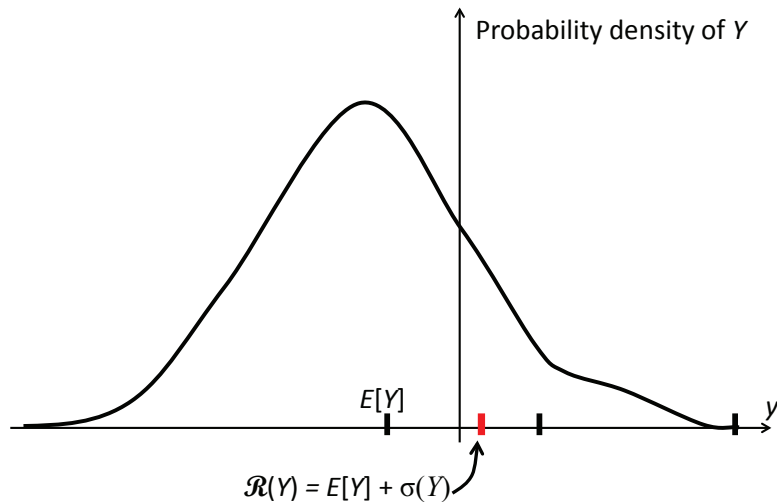
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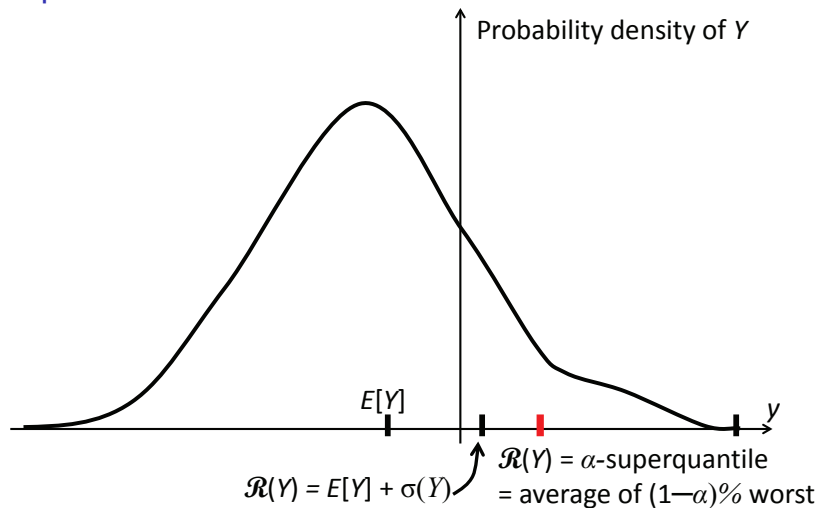
$\mathcal{R}(Y) = c$ when Y is identical to constant c

$\mathcal{R}(Y) > E[Y]$ when Y is not constant

Examples of risk measures

Probability density of Y

$E[Y]$

y

$\mathcal{R}(Y) = E[Y] + \sigma(Y)$

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Probability density of Y

$E[Y]$

$\mathcal{R}(Y) = \sup Y$

$\mathcal{R}(Y) = \alpha$-superquantile
= average of $(1-\alpha)\%$ worst

$\mathcal{R}(Y) = E[Y] + \sigma(Y)$

y
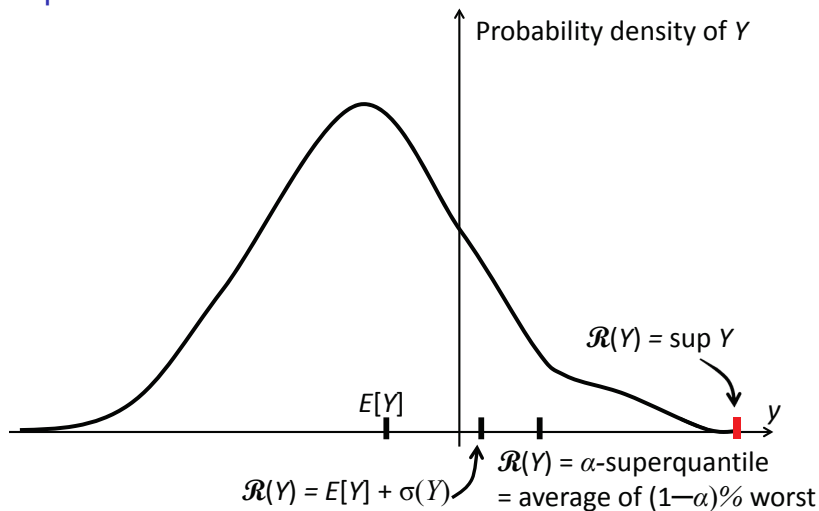
Risk-regret connection

Theorem:

Any regular measure of regret $\mathcal{V}$ constructs a regular measure of risk

$$\mathcal{R}(Y) = \min_{c_0 \in \mathbf{R}} \left\{ c_0 + \mathcal{V}(Y - c_0) \right\}.$$

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Measure of error $\mathcal{E} : \mathcal{L}^2 \rightarrow [0, \infty]$

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For example: $\mathcal{E}(Y) = E[Y^2]$

$$\mathcal{E}(Y) = E[\alpha \max\{0, Y\}/(1 - \alpha) + \max\{0, -Y\}]$$

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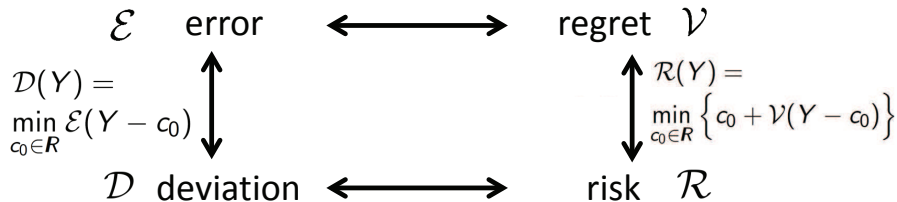
$\mathcal{D}(Y) > 0$ when Y is nonconstant

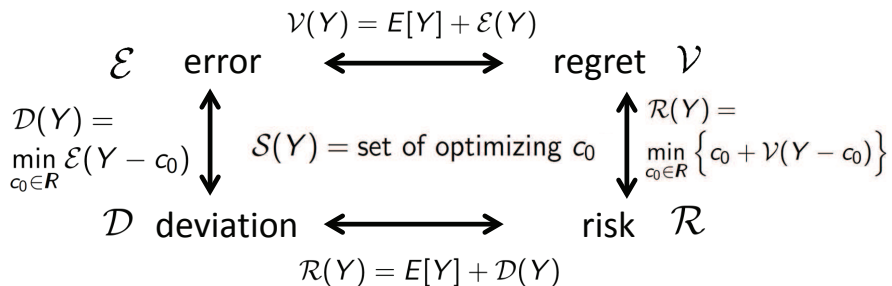# Deviation-error connection

Theorem:

Any regular measure of error $\mathcal{E}$ constructs a regular measure of deviation

$$\mathcal{D}(Y) = \min_{c_0 \in \mathbf{R}} \mathcal{E}(Y - c_0)$$

Connections

$$\mathcal{E} \quad \text{error} \quad \longleftrightarrow \quad \text{regret} \quad \mathcal{V}$$

$$\mathcal{D}(Y) = \min_{c_0 \in R} \mathcal{E}(Y - c_0)$$

$$\mathcal{R}(Y) = \min_{c_0 \in R} \left\{ c_0 + \mathcal{V}(Y - c_0) \right\}$$

$$\mathcal{D} \quad \text{deviation} \quad \longleftrightarrow \quad \text{risk} \quad \mathcal{R}$$

Connections

$$\mathcal{V}(Y) = E[Y] + \mathcal{E}(Y)$$

$\mathcal{E}$ error $\longleftrightarrow$ regret $\mathcal{V}$

$\mathcal{D}(Y) = \min\limits_{c_0 \in R} \mathcal{E}(Y - c_0)$ $\mathcal{S}(Y) = \text{set of optimizing } c_0$ $\mathcal{R}(Y) = \min\limits_{c_0 \in R} \left\{ c_0 + \mathcal{V}(Y - c_0) \right\}$

$\mathcal{D}$ deviation $\longleftrightarrow$ risk $\mathcal{R}$

$$\mathcal{R}(Y) = E[Y] + \mathcal{D}(Y)$$

Corresponding measures and statistics

Measures of residual risk: definition and properties

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With optimal c_0 and c:
$(c_0 + cE[X]) + (Y - [c_0 + cX]) \leq_{\mathcal{R}} \text{ residual risk}$

Motivation: wind prediction on day D
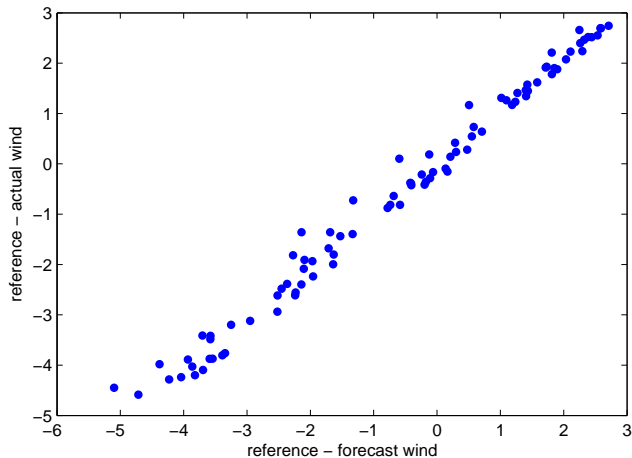
$Y = 3 - (\text{actual wind power}) = \text{shortfall}$

$X = 3 - (\text{forecast wind power}) = \text{legacy forecast}$

Preference-based predication

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- Same problem as faced by investor!

Measures of residual risk

For given random vector $X \in \mathcal{L}_n^2$ and regular measure of regret $\mathcal{V}$, a measure of residual risk $\mathcal{R}(\cdot|X) : \mathcal{L}^2 \to [-\infty, \infty]$ is defined by

$$\mathcal{R}(Y|X) = \inf_{c_0 \in \mathbf{R}, c \in \mathbf{R}^n} \left\{ c_0 + \langle c, E[X] \rangle + \mathcal{V}(Y - [c_0 + \langle c, X \rangle]) \right\}$$

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Interpretation: lowest $K \in \mathbf{R}$ such that

$$Y - [c_0 + \langle c, X \rangle] + [c_0 + \langle c, E[X] \rangle] \leq_{\mathcal{R}} K$$

Properties

Theorem:

Given $X \in \mathcal{L}_n^2$ and corresponding regular measures:

(i) $E[Y] \leq \mathcal{R}(Y|X) \leq \mathcal{R}(Y) \leq \mathcal{V}(Y)$.

Properties

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(ii) $\mathcal{R}(\cdot|X)$ is convex.

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   (i) $E[Y] \leq \mathcal{R}(Y|X) \leq \mathcal{R}(Y) \leq \mathcal{V}(Y)$.

  (ii) $\mathcal{R}(\cdot|X)$ is convex.

 (iii) If X is nondegenerate*, then $\mathcal{R}(\cdot|X)$ is closed and infimum attained.

*X is nondegenerate if $\langle c, X \rangle$ is a constant implies $c = 0$

Dual expression

Theorem:

For finite regular risk measure with conjugate $\mathcal{R}^*$ and risk envelope $\mathcal{Q} = \{Q \in \mathcal{L}^2 \mid \mathcal{R}^*(Q) < \infty\}$:

$$\mathcal{R}(Y|X) = \sup_{Q \in \mathcal{Q}} \left\{ E[QY] - \mathcal{R}^*(Q) \,\Big|\, E[QX] = E[X] \right\}$$

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Applications in optimization under stochastic ambiguity

Alternative perspective: linear regression

Find regression coefficients c_0 and c that

$$\text{minimize } \mathcal{E}(Y - [c_0 + \langle c, X \rangle])$$

What will we obtain for "nonstandard" error measures?

Connection regression and residual risk

Regression Problem:
$$\min_{c_0, c} \mathcal{E}(Y - [c_0 + \langle c, X \rangle])$$

Residual Risk Problem:
$$\min_{c_0, c} \left\{ c_0 + \langle c, E[X] \rangle + \mathcal{V}(Y - [c_0 + \langle c, X \rangle]) \right\}$$

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Given $X \in \mathcal{L}_n^2$ and corresponding regular measures:

(i) Optimal solution sets are identical.

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Given $X \in \mathcal{L}_n^2$ and corresponding regular measures:

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(ii) They are closed, convex, and if X is nondegenerate, then also nonempty.

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Theorem:

Given $X \in \mathcal{L}_n^2$ and corresponding regular measures:

(i) Optimal solution sets are identical.

(ii) They are closed, convex, and if X is nondegenerate, then also nonempty.

(iii) They are bounded if the residual risk is finite and X is nondegenerate.

Application to surrogate models

Learn from low-fidelity simulation: drag/lift estimation
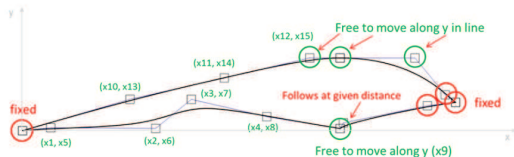
Y drag-to-lift ratio; costly realization (4 hours on 8 cores)
X approx. ratio; inexpensive realizations (5 sec on 1 core)

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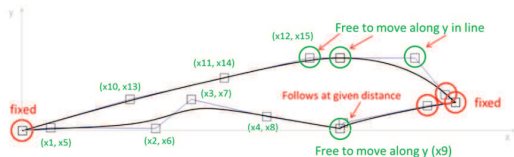
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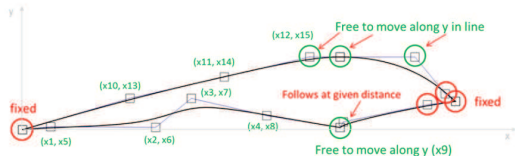
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Find $c_0 + cX$ such that Y safely $\leq c_0 + cX$
$\mathcal{R}(Y) \leq \mathcal{R}(c_0 + cX)$, with 0.8-superquantile risk

Distribution of drag/lift

Randomness due to manufacturing tolerance (689 "scenarios")
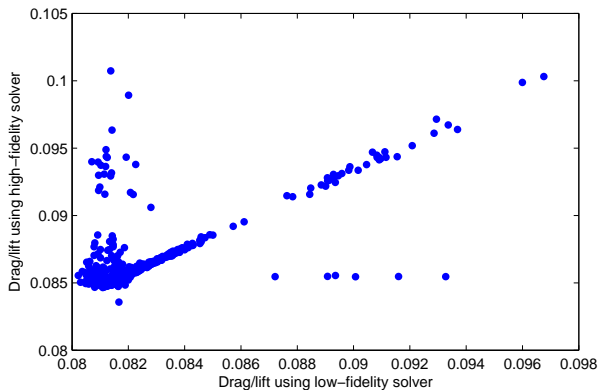
Mean $E[Y] = 0.0864$; 0.8-superquantile $\mathcal{R}(Y) = 0.0901$

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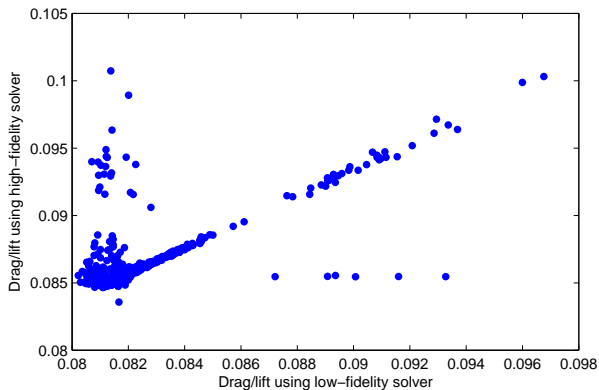
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Want to bound $\mathcal{R}(Y)$ from above cheaply
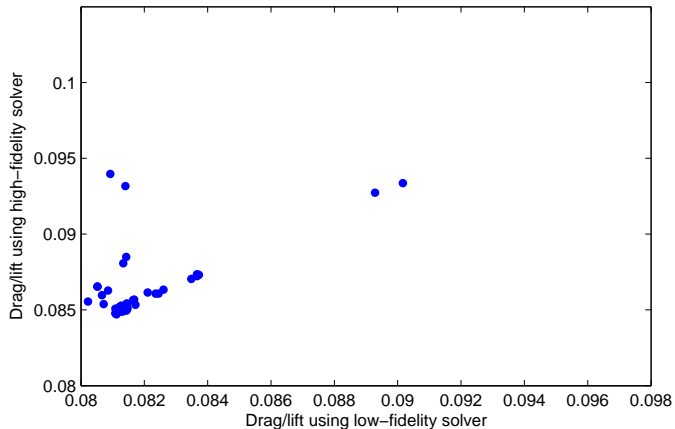
Risk-tuned surrogate estimation

For positively homogeneous regular measure of risk $\mathcal{R}$,

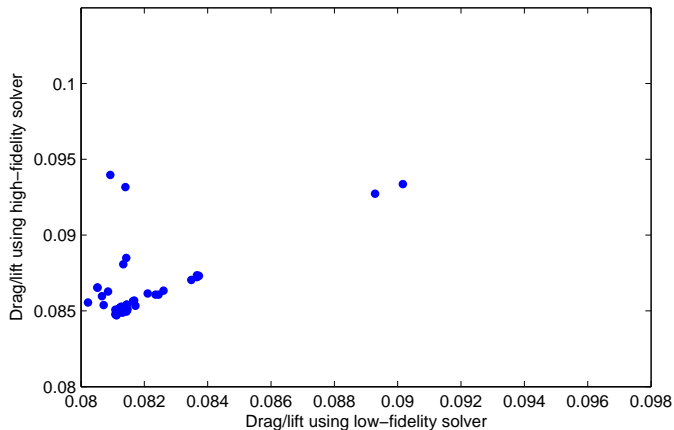$$\text{the model } c_0 + \langle c, X \rangle \text{ of } Y,$$

with c obtained by corresponding regression and
$c_0 = \mathcal{R}(Y - \langle c, X \rangle)$, satisfies

$$\mathcal{R}(Y) \leq \mathcal{R}(c_0 + \langle c, X \rangle)$$
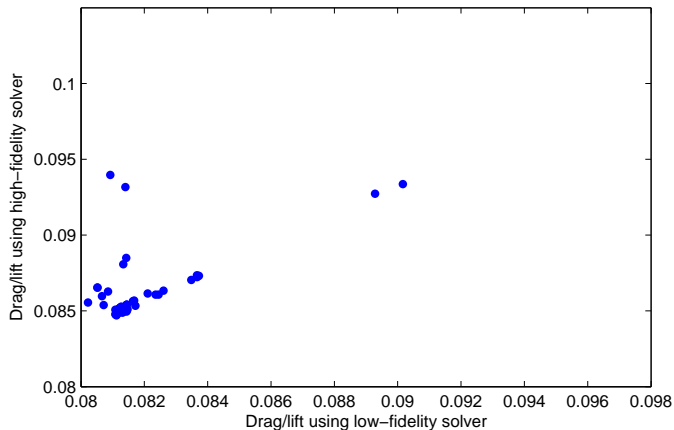
50 training data points

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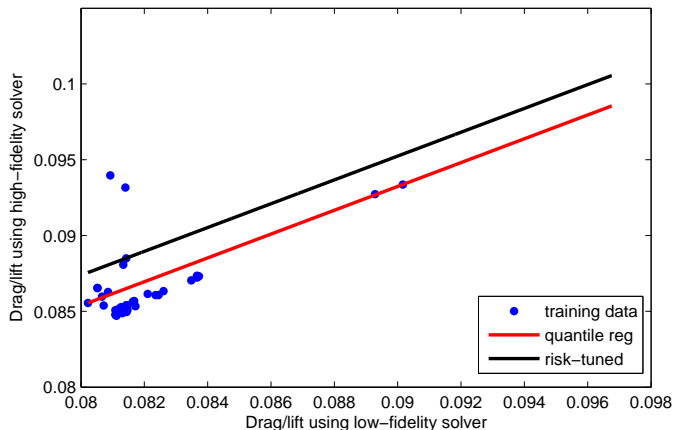Quantile regression/residual risk problem $\longrightarrow c = 0.7859$

50 training data points

Quantile regression/residual risk problem $\longrightarrow c = 0.7859$
Following theorem $\longrightarrow c_0 = 0.0245$

Illustration of fit

Illustration of fit

$\mathcal{R}(c_0 + cX) = 0.0918$ (mean 0.0912, st.dev. 0.0016)

Recall: $\mathcal{R}(Y) = 0.0901$

Summary

Connecting estimation and decision making (risk-tuning)

Enabling construction of risk-averse, preference-driven data tools

Applications in "risk-averse" regression and robust optimization

Reference

R.T. Rockafellar and J.O. Royset,
"Measures of Residual Risk with Connections to Regression, Risk
Tracking, Surrogate Models, and Ambiguity,"
SIAM J. Optimization, to appear

http://faculty.nps.edu/joroyset