

# Martingale Optimal Transport and Robust Hedging

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Limoges, May 22, 2015

# Outline

- 1 Optimal Transport and Model-free hedging
  - The Monge-Kantorovitch optimal transport problem
  - Financial interpretation
- 2 Martingale Transport Problem
  - Formulation and duality
  - Optimal semi-static strategy and quasi-sure formulation
- 3 Geometry of Optimal Transport Measure
  - Back to Standard Optimal Transport
  - Martingale Monotonicity Condition
  - Martingale Version of the 1-dim Brenier Theorem

M É M O I R E  
S U R L A  
T H É O R I E D E S D É B L A I S  
E T D E S R E M B L A I S.

Par M. M O N G E.

LORSQU'ON doit transporter des terres d'un lieu dans un autre, on a coutume de donner le nom de *Déblai* au volume des terres que l'on doit transporter, & le nom de *Remblai* à l'espace qu'elles doivent occuper après le transport.

Le prix du transport d'une molécule étant, toutes choses d'ailleurs égales, proportionnel à son poids & à l'espace qu'on lui fait parcourir, & par conséquent le prix du transport total devant être proportionnel à la somme des produits des molécules multipliées chacune par l'espace parcouru, il s'en suit que le déblai & le remblai étant donnés de figure & de position, il n'est pas indifférent que telle molécule du déblai soit transportée dans tel ou tel autre endroit du remblai, mais qu'il y a une certaine distribution à faire des molécules du premier dans le second, d'après laquelle la somme de ces produits sera la moindre possible, & le prix du transport total fera un *minimum*.

# The problem of “Déblais et Remblais” (Monge 1781)

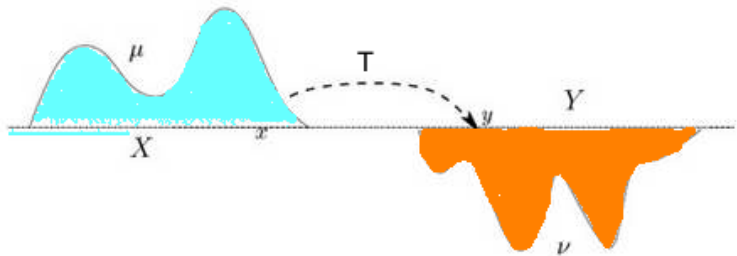


Figure: Mass transport. Ref : C. Villani

## Analytic formulation (Monge 1781)

- Initial distribution : probability measure  $\mu$  on  $\mathcal{X}$
- Final distribution : probability measure  $\nu$  on  $\mathcal{Y}$

**Problem** : find an optimal transference plan  $T^*$

$$P^M := \sup_{T \in \mathcal{T}(\mu, \nu)} \int c(x, T(x)) \mu(dx)$$

where  $\mathcal{T}(\mu, \nu)$  of all maps  $T : x \mapsto y = T(x)$  such that

$$\nu = \mu \circ T^{-1}$$

# Randomization of mass transfert (Kantorovich 1942)

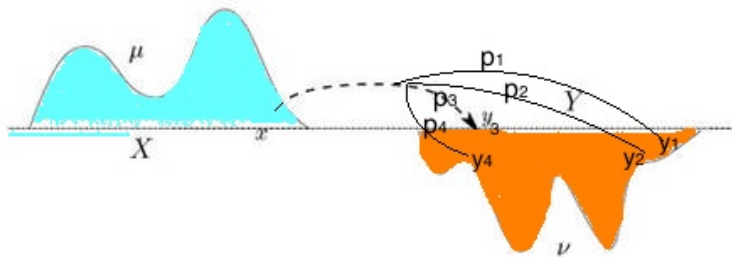


Figure: Randomized mass transport

# Probabilistic formulation (Kantorovich 1942)

Randomization of transference plans :

$$\bar{P}^K := \sup_{\mathbb{P} \in \mathcal{P}(\mu, \nu)} \mathbb{E}^{\mathbb{P}} [c(X, Y)]$$

where  $\Omega = \mathcal{X} \times \mathcal{Y}$ ,  $X(x, y) = x$ ,  $Y(x, y) = y$ , and

$$\mathcal{P}(\mu, \nu) := \{ \mathbb{P} \in \text{Prob}_{\Omega} : X \sim_{\mathbb{P}} \mu \text{ and } Y \sim_{\mathbb{P}} \nu \}$$

# Kantorovich duality

Duality in linear programming, Legendre-Fenchel duality...

$$\mathcal{D}^0 := \inf_{(\varphi, \psi) \in \mathcal{D}^0} \{ \mu(\varphi) + \nu(\psi) \}$$

$$\mathcal{D}^0 := \{ (\varphi, \psi) \in \mathbb{L}^1(\mu) \times \mathbb{L}^1(\nu) : \varphi \oplus \psi \geq c \}$$

where  $\mu(\varphi) := \int \varphi d\mu$ ,  $\nu(\psi) := \int \psi d\nu$ , and

$$\varphi \oplus \psi(x, y) := \varphi(x) + \psi(y), \quad x \in \mathcal{X}, y \in \mathcal{Y}$$

**No reference probability measure on the product space**



# The duality

## Theorem

*Let  $c \in USC$  and bounded from above. Then  $D^0 = P^K$ , and existence holds for  $P^K$*

- Duality extends to  $c$  measurable (Kellerer '84)
- Existence for the dual problem holds under some conditions on  $\mathcal{X}, \mathcal{Y}$  and  $c$

# Financial interpretation

- $X \sim \mu$  and  $Y \sim \nu$  prices of two assets at time 1
- $\mu$  and  $\nu$  identified from market prices of call options :

$$C_{\mu}(K) = \int (x - K)^+ \mu(dx), \quad C_{\nu}(K) = \int (y - K)^+ \nu(dy), \quad K \geq 0$$

Breeden-Litzenberger 1978 :  $\mu = C_{\mu}''$  and  $\nu = C_{\nu}''$

- $\varphi(X)$  Vanilla position in  $X$  with market price  $\mu(\varphi)$
- $\psi(Y)$  ...  $Y$  ...  $\nu(\psi)$

## Financial interpretation : no reference probability

- $c(X, Y)$  payoff of derivative security
- **Robust static** hedging strategies for the derivative  $c(X, Y)$  :

$$\mathcal{D}^0 := \{(\varphi, \psi) \in \mathbb{L}^1(\mu) \times \mathbb{L}^1(\nu) : \varphi \oplus \psi \geq c\}$$

**Robust superhedging** cost of  $c(X, Y)$  is the Kantorovitch dual :

$$D^0 = \inf_{(\varphi, \psi) \in \mathcal{D}^0} \{\mu(\varphi) + \nu(\psi)\}$$

The primal Monge-Kantorovitch problem is :

$$P^K = \sup_{\mathbb{P} \in \mathcal{P}(\mu, \nu)} \mathbb{E}^{\mathbb{P}}[c(X, Y)]$$

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## MARTINGALE OPTIMAL TRANSPORT

initiated by Pierre Henry-Labordère, Preprints Soc. Gén.

- Beiglböck, Henry-Labordère & Penkner
- Galichon, Henry-Labordère & NT

# One asset observed at two future dates

Our interest now is on the case where

$$X = X_0 \quad \text{and} \quad Y = X_1$$

are the prices of **the same asset** at two **future dates 0 and 1**

Interest rate is reduced to zero

This setting introduces a new feature :

**possibility of dynamic trading the asset between times 0 and 1**

# Superhedging problem $\equiv$ Kantorovitch dual

Robust super hedging problem naturally formulated as :

$$D(\mu, \nu) := \inf_{(\varphi, \psi, h) \in \mathcal{D}} \{ \mu(\varphi) + \nu(\psi) \}$$

where

$$\mathcal{D} := \{ (\varphi, \psi, h) \in \mathbb{L}^1(\mu) \times \mathbb{L}^1(\nu) \times \mathbb{L}^0 : \varphi \oplus \psi + h^{\otimes} \geq c \}$$

where

$$\varphi \oplus \psi(x, y) := \varphi(x) + \psi(y) \quad \text{and} \quad h^{\otimes}(x, y) := h(x)(y - x)$$

# The Martingale Optimal Transport Problem

Theorem (Beiglböck, Henry-Labordère, Penkner)

Assume  $c \in USC$  and bounded from above. Then  $P = D$ , and existence holds for  $P(\mu, \nu)$ .

where the dual problem is :

$$P(\mu, \nu) := \sup_{\mathbb{P} \in \mathcal{M}(\mu, \nu)} \mathbb{E}^{\mathbb{P}} [c(X, Y)]$$

with

$$\mathcal{M}(\mu, \nu) := \{ \mathbb{P} \in \mathcal{P}(\mu, \nu) : \mathbb{E}^{\mathbb{P}} [Y|X] = X \}$$

Strassen '65 :  $\mathcal{M}(\mu, \nu) \neq \emptyset$  iff  $\mu \preceq \nu$ , i.e.  $\mu(g) \leq \nu(g) \forall g$  convex



## Existence of optimal hedge does not hold in general

- There are easy examples where there is no optimal robust superhedging strategy, even for bounded  $c$ , bounded support...
- Let  $\mu = \nu$ , then  $\mathcal{M}(\mu, \mu) = \{\mathbb{P}^*\}$  where  $Y = X$ ,  $\mathbb{P}^*$ -a.s.

- The martingale transport problem is degenerate

$$P(\mu, \mu) = \mathbb{E}^\mu [c(X, X)]$$

- The derivative is in fact defined by  $c(X, X)$ , primal problem carries no information about  $c(x, y)$  outside the diagonal  $y = x$
- One is only interested in hedging along the diagonal

$$\varphi(X) + \psi(X) + h(X)(X - X) \geq c(X, X), \quad \mu - \text{a.s.}$$



# Duality under more general payoff functions

- The condition  $c \in \text{USC}$  is not innocent...
- Consider the following example of LSC payoff

$$c(x, y) := \mathbb{1}_{\{x \neq y\}}, \quad x, y \in [0, 1] \times [0, 1]$$

Let  $\mu = \nu = \text{Lebesgue measure on } [0, 1]$ . Then

- $\mathcal{M}(\mu, \mu) = \{\mathbb{P}^*\}$  uniform distribution on the diagonal of the square  $[0, 1]^2$
- Then  $P(\mu, \mu) = 0$
- However, we may prove that  $D(\mu, \mu) = 1!$

# Quasi-sure robust superhedging

## Definition

$\mathcal{M}(\mu, \nu)$ -q.s. (quasi surely) means  $\mathbb{P}$ -a.s. for all  $\mathbb{P} \in \mathcal{M}(\mu, \nu)$

- The quasi-sure robust superhedging cost

$$D^{qs} := \inf_{(\varphi, \psi, h) \in \mathcal{D}^{qs}} \{ \mu(\varphi) + \nu(\psi) \}$$

$$\mathcal{D}^{qs} := \{ (\varphi, \psi, h) : \varphi \oplus \psi + h^{\otimes} \geq c, \mathcal{M}(\mu, \nu) - \text{q.s.} \}$$

is more natural...

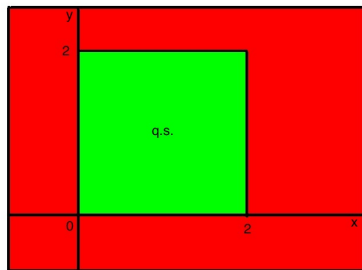
- Then,  $D(\mu, \nu) \geq D^{qs}(\mu, \nu) \geq P(\mu, \nu)$

so if the duality  $P = D$  holds, it follows that  $D = D^{qs}$

# Pointwise versus Quasi-sure superhedging I

Suppose  $\text{Supp}(\mu) = [0, 2] = \text{Supp}(\nu) = [0, 2]$ , then

- $\mathcal{M}(\mu, \nu)$ -q.s. only involves the values  $(x, y) \in [0, 1]^2$
- Pointwise superhedging involves all values  $(x, y) \in \mathbb{R}^2$

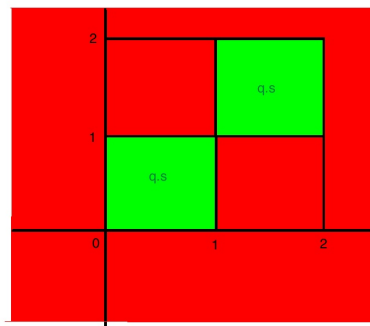


## Pointwise versus Quasi-sure superhedging II

Suppose  $\text{Supp}(\mu) = \text{Supp}(\nu) = [0, 2]$ , and  $C_\mu(1) = C_\nu(1)$

$$\mathbb{E}[(X - 1)^+] = \mathbb{E}[(Y - 1)^+] \leq \mathbb{E}[(X - 1)^+]$$

by Jensen's inequality, and then  $\{X \geq 1\} = \{Y \geq 1\}$



# Duality and existence under quasi-sure robust superhedging

Theorem (Beiglböck, Nutz & NT '15)

Let  $\mu \preceq \nu$  and  $c \geq 0$  measurable. Then

$$P(\mu, \nu) = D^{qs}(\mu, \nu)$$

and existence holds for both  $P$  and  $D^{qs}$

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# Optimizers of the Transport problem and its dual

- Optimal transport problem  $P^K(\mu, \nu)$  has a solution  $\mathbb{P}^*$ , this is a consequence of the compactness of  $\mathcal{P}(\mu, \nu)$

Any characterization of  $\mathbb{P}^*$  ??

- This provides a way to determine the solution of the dual...



# Geometry of optimal transport plans : cyclic monotonicity

Optimality of a transport plan is a property of its support...

## Theorem

For an optimal transport plan  $\mathbb{P}^*$ , there exists  $\Gamma \subset \mathcal{X} \times \mathcal{Y}$  :

- $\mathbb{P}^*[\Gamma] = 1$ ,
- and for all finite subset  $(x_i, y_i)_{i \leq n} \subset \Gamma$  :

$$\sum_{i=1}^n c(x_i, y_i) \geq \sum_{i=1}^n c(x_i, y_{i+1}) \quad \text{with} \quad y_{n+1} = y_1$$

(necessary and sufficient, under slight conditions)

## Back to the original Monge formulation

- $P^K \geq P^M$  : Kantorovitch formulation  $\equiv$  relaxation of Monge one

Theorem (Rachev & Rüschendorf)

$\mu$  without atoms,  $c \in C^1$  with  $c_{xy} > 0$  (Spence-Mirrlees/Twist condition). Then there is a unique optimal transference plan :

$$\mathbb{P}^*(dx, dy) = \mu(dx) \delta_{\{T^*(x)\}}(dy) \quad \text{with} \quad T^* = F_\nu^{-1} \circ F_\mu$$

Consequently  $P^M = P^K$ , and  $T^*$  solves both problems.

- $T^*$  : monotone rearrangement, Fréchet-Hoeffding coupling
- Extension to  $\mathbb{R}^d$  (Brenier) :  $\mathbb{P}^*$  concentrated on a graph defined by the gradient of some  $c$ -convex function

# On the Spence-Mirrlees condition

The solution of the Kantorovitch optimal transportation problem

$$P^K := \sup_{\mathbb{P} \in \mathcal{P}(\mu, \nu)} \int c(x, y) \mathbb{P}(dx, dy)$$

is not modified by the change of performance criterion :

$$c(x, y) \longrightarrow \hat{c}(x, y) := c(x, y) + a(x) + b(y)$$

Spence-Mirrlees condition  $c_{xy} > 0$  stable by this transformation

## Martingale monotonicity condition

Optimality of martingale transport is also a property of its support...

Theorem (Beiglböck & Juillet '12, Zhev '14)

Let  $\mathbb{P}^*$  be a solution of  $P(\mu, \nu)$ . Then

- there exists  $\Gamma \subset \Omega$ ,  $\mathbb{P}^*[\Gamma] = 1$  such that
- for all  $\mathbb{P}_0^*$  with  $\text{Supp}(\mathbb{P}_0^*) = \{\omega_1, \dots, \omega_N\} \subset \Gamma$ , we have :

$$\mathbb{E}^{\mathbb{P}_0^*} [c(X, Y)] \geq \mathbb{E}^{\mathbb{P}} [c(X, Y)]$$

whenever

$$\mathbb{P} \circ X^{-1} = \mathbb{P}_0^* \circ X^{-1}, \quad \mathbb{P} \circ Y^{-1} = \mathbb{P}_0^* \circ Y^{-1}$$

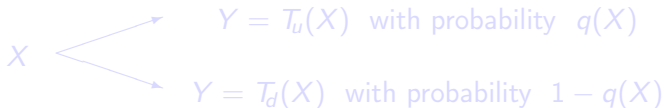
and  $\mathbb{E}^{\mathbb{P}}[Y|X] = \mathbb{E}^{\mathbb{P}_0^*}[Y|X]$

## Worst Case Financial Market – Brenier Theorem

- Solution  $\mathbb{P}^* \in \mathcal{M}(\mu, \nu)$  exists. Is there an optimal transfert map, i.e. optimal transport of  $\mu$  to  $\nu$  through a map  $T^*$ ?

**NO**, unless  $\mu = \nu$ !

Is there a transference plan along a minimal randomization ?

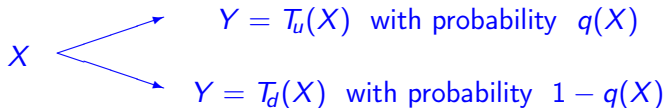


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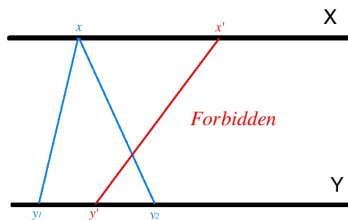


# Left-monotone martingale transport

## Definition (Beiglböck & Juillet 2012)

$\mathbb{P} \in \mathcal{M}(\mu, \nu)$  is **left-monotone** if  $\mathbb{P}[(X, Y) \in \Gamma] = 1$ , for some  $\Gamma \subset \mathbb{R} \times \mathbb{R}$ , and

for all  $(x, y_1), (x, y_2), (x', y') \in \Gamma$ :  $x < x' \implies y' \notin (y_1, y_2)$



# Existence and uniqueness of left-monotone martingale transport

Theorem (Beiglböck, Henry-Labordère & NT '15)

Let  $\mathbb{P}^* \in \mathcal{M}(\mu, \nu)$  be solution of  $P(\mu, \nu)$ . If  $c_x$  strictly convex in  $y$ ,  $\mu$ -a.e.  $x$ , then  $\mathbb{P}^*$  is left-monotone

Theorem (Beiglböck & Juillet '12, Beiglböck, Henry-Labordère & NT '15)

Assume  $\mu$  has no atoms. Then, there is a unique left-monotone  $\mathbb{P} \in \mathcal{M}(\mu, \nu)$  with distribution concentrated on two graphs

Henry-Labordère & NT '13 provide an explicit description of

- the left-monotone transport plan
- and corresponding semi-static robust superhedging strategy



# The martingale version of the Spence-Mirrlees condition

... is  $c_{xyy} > 0$

- Notice that the solution of the Martingale Transport problem is not altered by the change of payoff :

$$c(x, y) \longrightarrow \hat{c}(x, y) := c(x, y) + a(x) + b(y) + h(x)(y - x)$$

- $\hat{c}_{xyy} = c_{xyy}$

## Concluding remarks

- Extension to **finite discrete-time**, possibly **finitely-many marginals** constraints
- **Continuous-Time Martingale Transport** : substitute for the return from a dynamic hedge  $h^\otimes$  is the stochastic integral

$$\int_0^T H_s dX_s$$

But **without reference probability ??**  $\implies$  Two viewpoints

- **Pointwise definition** : restrict  $H$  to have finite variation, then :

$$\int_0^T H_s dX_s := \int_0^T X_s dH_s + H_T X_T - H_0 X_0$$

- **Quasi-sure definition** : under any  $\mathbb{P} \in \mathcal{M}(\mu, \nu)$ ,  $\int_0^T H_s dX_s$  is defined by the standard stochastic analysis...

THANK YOU FOR YOUR  
ATTENTION