Variational Analysis, Terry's fest, Limoges (France), Spring 2015

A tale about bifunctions

Roger J-B Wets
Mathematics
University of California, Davis

Convex Optimization

Convex minimization problem:

$\min f_0(x)$ such that $x \in X \subset \mathbb{R}^n$

$f_i(x) \leq 0 \ , i = 1, \dots, s, \quad f_i(x) = 0 \ , i = s+1, \dots, m$

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$$F(\cdot, 0) \in \left\{ F(\cdot, u), u \in U \subset \mathbb{R}^m \right\}$$

$$u \mapsto S(u) = \left\{ x \in X \,\middle|\, f_i \leq 0,\ f_i = 0 \right\},$$ feasibility mapping

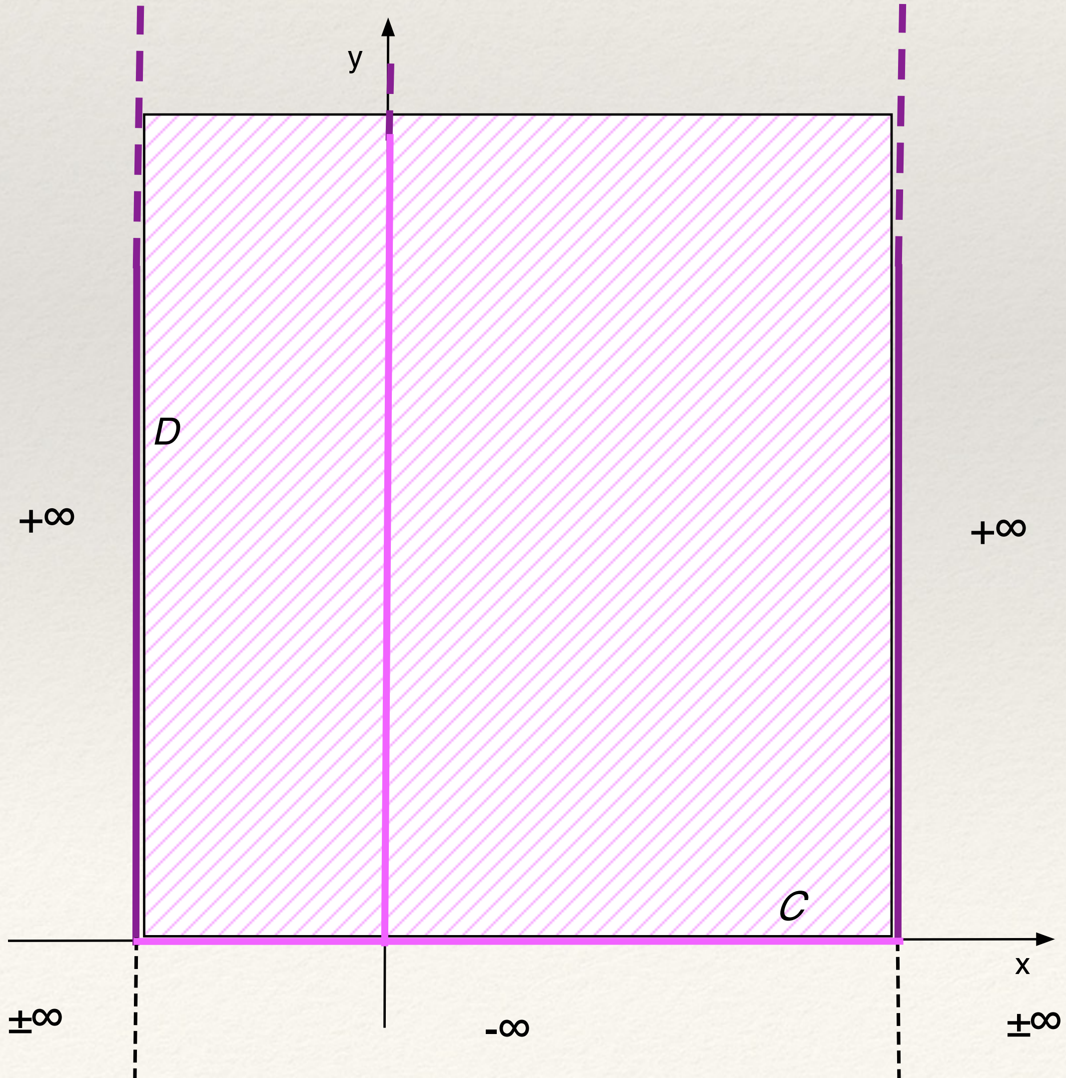
$$F(u, x) = \begin{cases} f_0(x) & \text{when } x \in S(u) \\ \infty & \text{when } x \in S(u) \end{cases}$$

$F : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$, set $U = \mathbb{R}^m$, convex bifunction

Saddle Functions: Part VII " Convex Analysis"

$F(x, u)$ a convex bifunction on $\mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$

$$L(x, y) = \inf_u \{\langle y, u\rangle - F(x, u)\}$$ Lagrangian



$$C = X, \quad D = \mathbb{R}^s_+ \times \mathbb{R}^{m-s}$$

$$L(x, y) = \begin{cases} f_0(x) + \sum_{i=1}^n y_i f_i(x) & \text{if } x \in C, y \in D \\ +\infty & \text{if } y \in D,\ x \notin C \\ -\infty & \text{if } y \notin D \end{cases}$$

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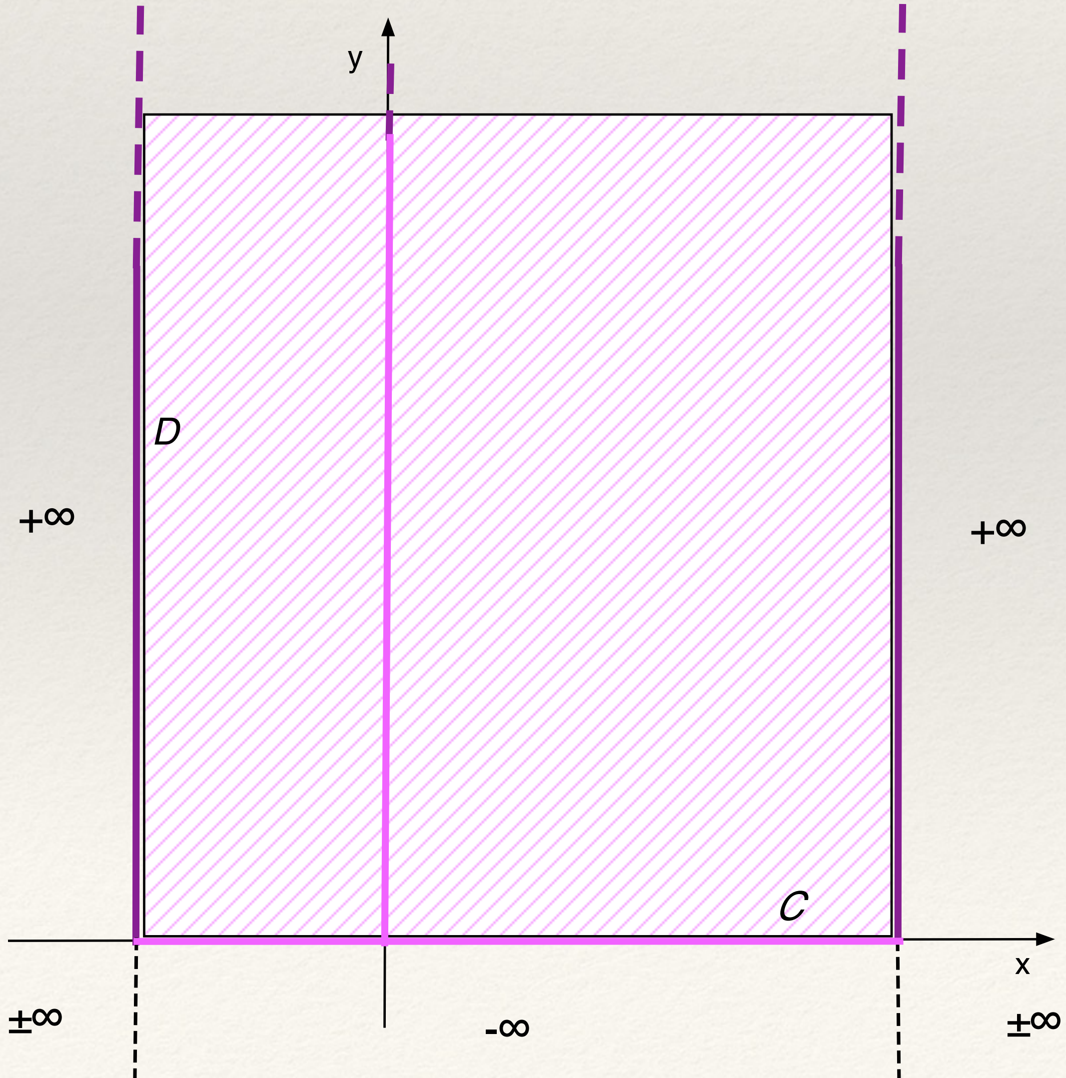

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early 80's: with Hedy Attouch, + Dominique Azé (later)

Convergence of Saddle Functions

$F, F^\nu : X \times Y \to \overline{\mathbb{R}}, \quad (X, \tau), (Y, \sigma)$ metrizable

Epi/hypo-convergence (∼ hypo/epi-convergence)

$\forall\, x^\nu \to_\tau x, \exists\, y^\nu \to_\sigma y$: $F(x,y) \leq \liminf_\nu F^\nu(x^\nu, y^\nu)$

$\forall\, y^\nu \to_\sigma y, \exists\, x^\nu \to_\sigma x$: $F(x,y) \geq \limsup_\nu F^\nu(x^\nu, y^\nu)$

$\Rightarrow$ saddle pnts of F^ν "converge" to saddle pnts of F

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Limit functions: ν filter: $\mathcal{H}$

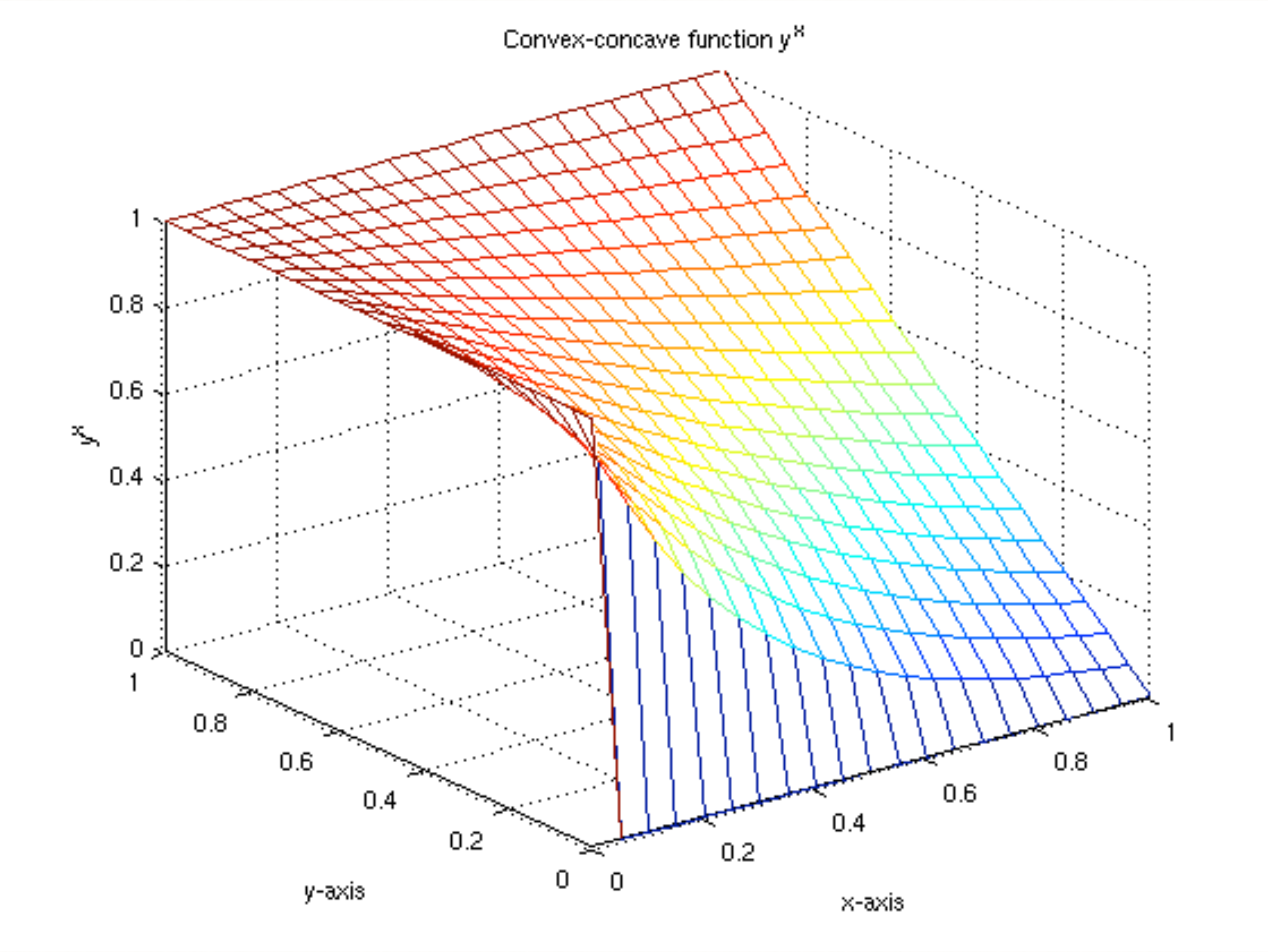
e-limit: $l_e F^\nu = \sup_{U \in \mathcal{N}_\tau(x)} \inf_{V \in \mathcal{N}_\sigma(y)} \inf_{H \in \mathcal{H}} \sup_{\nu \in H} \sup_{v \in V} \inf_{u \in U} F^\nu(u,v)$

h-limit: $l_h F^\nu = \inf_{V \in \mathcal{N}_\sigma(y)} \sup_{U \in \mathcal{N}_\tau(x)} \sup_{H \in \mathcal{H}} \inf_{\nu \in H} \sup_{v \in V} \inf_{u \in U} F^\nu(u,v)$

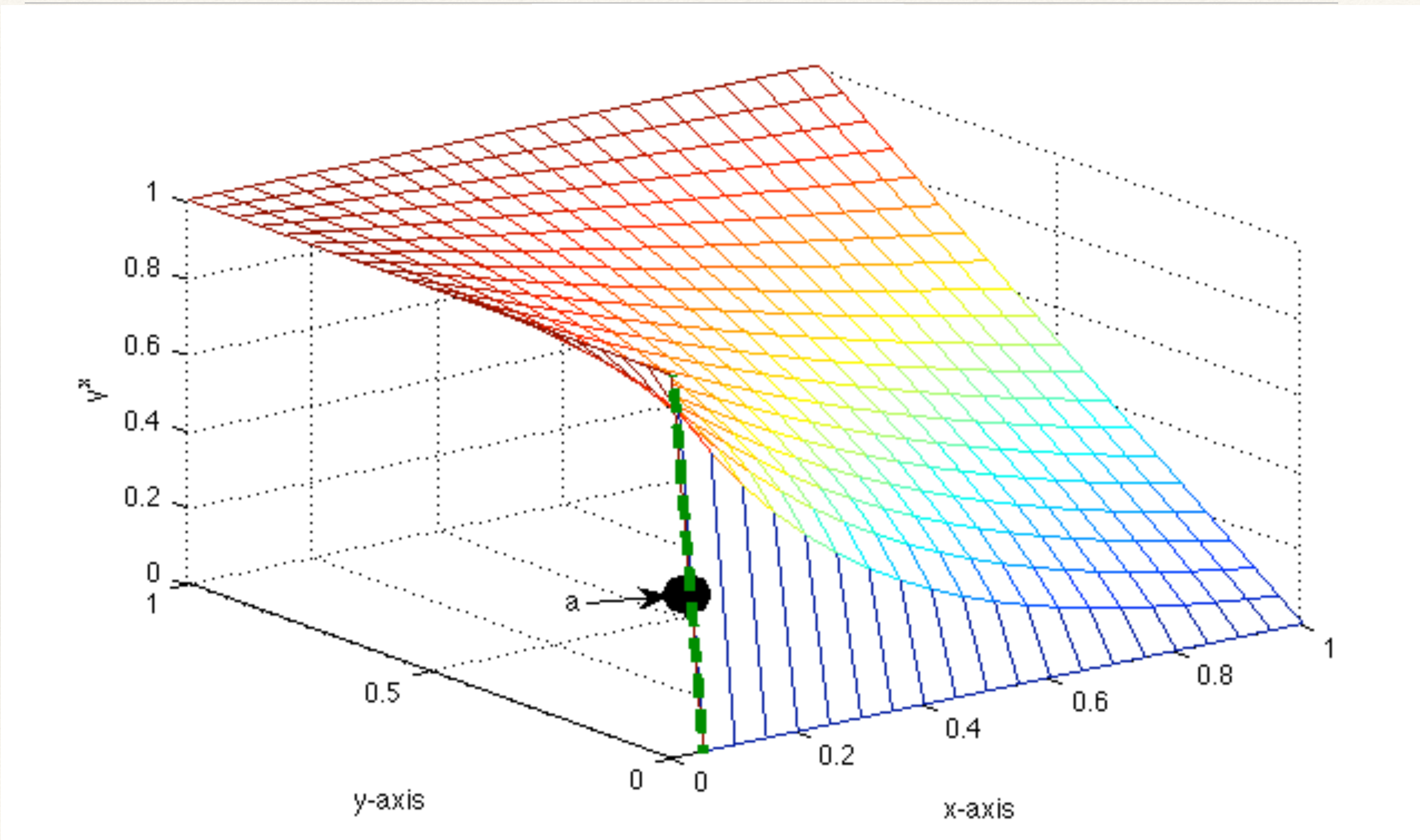
F an <u>epi/hypo-limit</u> $\iff$ $l_e F^\nu \le F \le l_h F^\nu$

Epi/hypo-convergence: not a Hausdorff topology ! !

$F^\nu = y^x$ on $[0,1]$ with $\pm\infty$ convex-concave extensions



epi/hypo-limit: $F = F^{\nu}$ on $[0,1]^2 \setminus \{0,0\}$, $F(0,0) = a \in [0,1]$



Equivalence Classes!

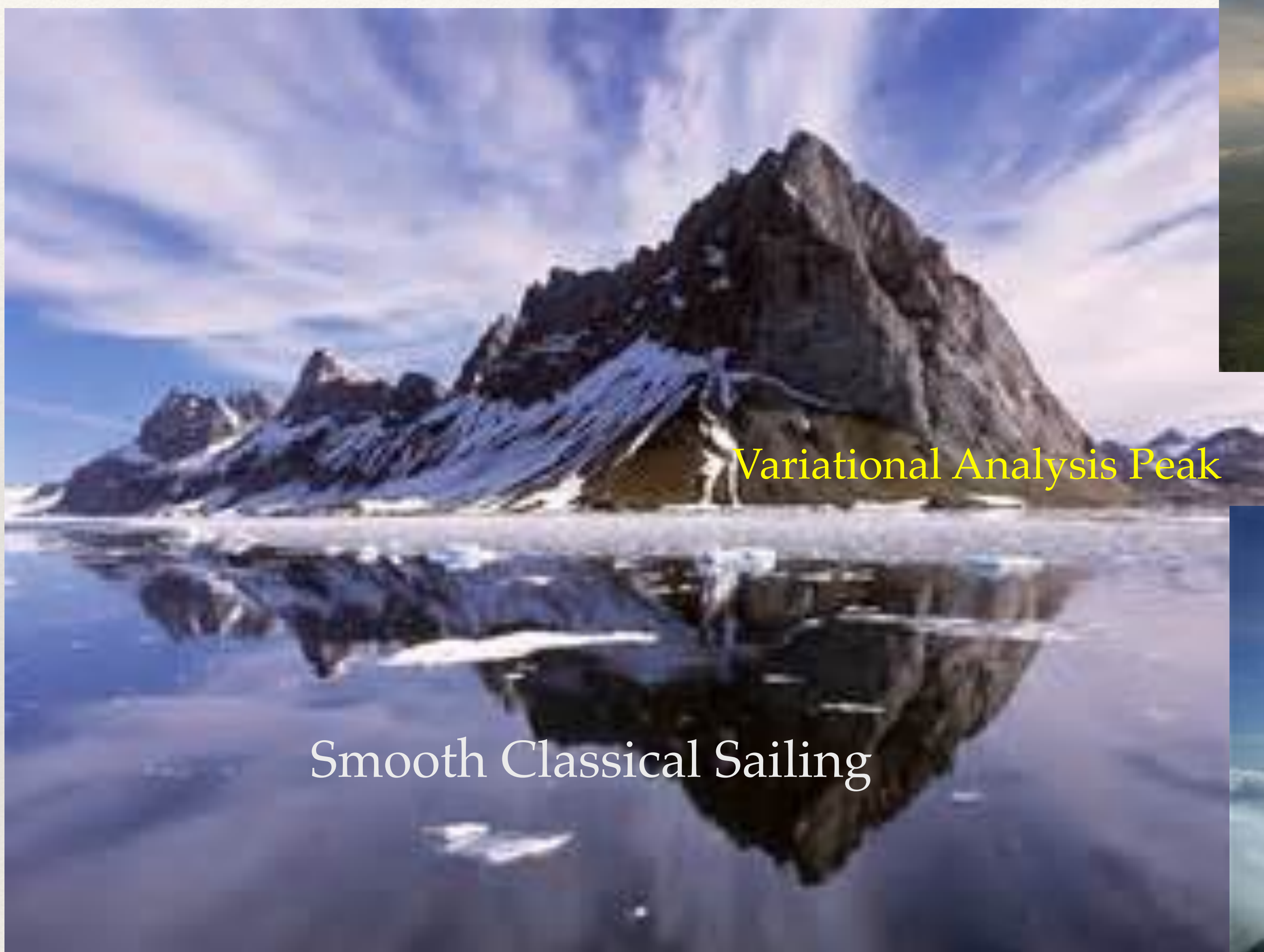
F, G are equivalent $\mathrm{cl}_x\, F = \mathrm{cl}_x\, G$ and $\mathrm{cl}_y\, F = \mathrm{cl}_y\, G$ 1970

convergence of equivalence classes

1995 notes (for us) shows that epi/hypo convergence

- is consistent with
- and can be formulated in terms of equivalence classes


Variational Analysis Peak
Smooth Classical Sailing



Celebrating your 50th anniversary of your 30th birthday

2nd-order Subderivatives

- attempt at integrating subdifferentiability of 2nd order with epi/hypo-limits (RTR)
- obstacle: lack of uniqueness!
- Chi Do


a bleak picture

A new paradigm

- Jean-Pierre Aubin, office neighbor at IIASA, working with Ivar Ekeland on "Applied Nonlinear Analysis"
- from saddle point to convergence of just MaxInf-points
- Lopsided Convergence for extended-real valued bifunctions (1983, CRAS)

Lopsided Convergence

$F, F^\nu : X \times Y \to \overline{\mathbb{R}}, \quad (X, \tau), (Y, \sigma)$ metrizable

Epi/hypo-convergence saddle points converge

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“only”

Lopsided convergence MaxInf-points converge

$\forall\, x^\nu \to_\tau x, \exists\, y^\nu \to_\sigma y$: $F(x,y) \le \liminf_\nu F^\nu(x^\nu, y^\nu)$

$\exists\, x^\nu \to_\tau x, \forall\, y^\nu \to_\sigma y,$: $F(x,y) \ge \limsup_\nu F^\nu(x^\nu, y^\nu)$

Equilibrium as a MinSup

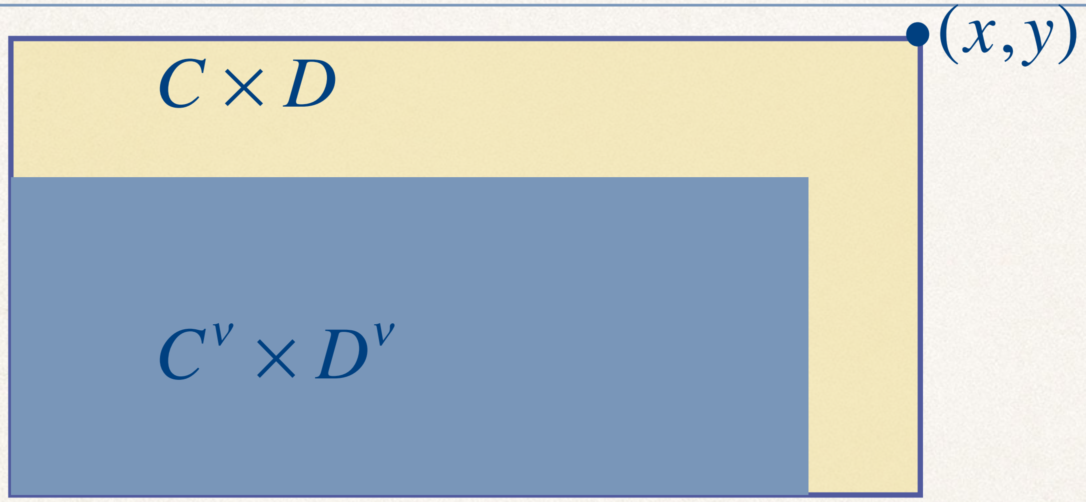
Actually MaxInf

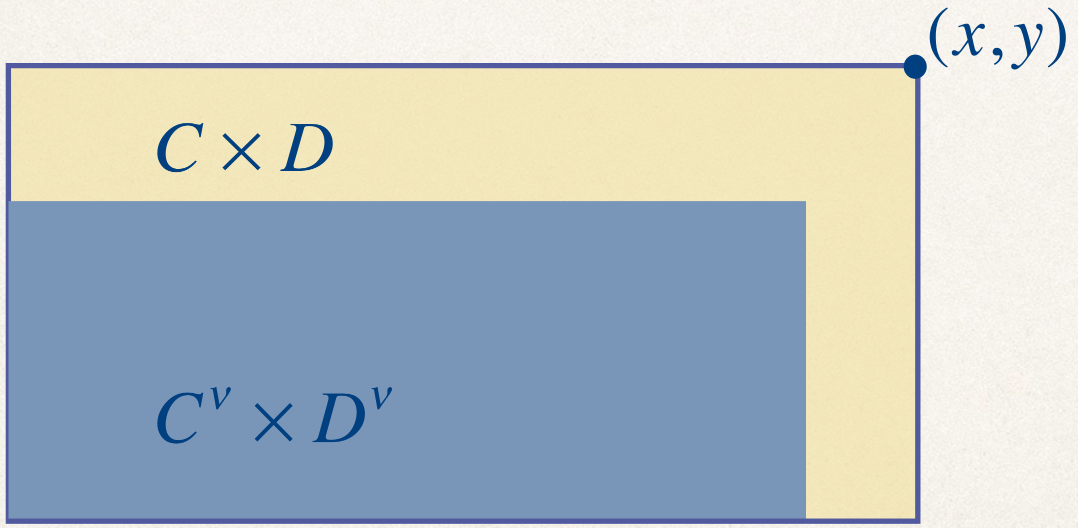
- $\forall a \in A$: $d_a(p) \in \arg\max\left\{u_a(x_a) \,\middle|\, \langle p, x_a\rangle \le \langle p, e_a\rangle\right\}$
  $s(p) = \sum_a (e_a - d_a(p))$ excess supply
- find $\bar{p} \in \Delta$ (unit simplex) so that $s(\bar{p}) \ge 0$
- Walrasian: $W(p,q) = \langle q, s(p)\rangle$ Ky Fan fcn
- $\bar{p} \in \operatorname{maxinf} W, W(\bar{p}, \bullet) \ge 0 \Leftrightarrow s(\bar{p}) \ge 0$
- conditions: $u_a^\nu \to_{hypo} u_a,\ e_a^\nu \to e_a \in \text{int dom u}_a \Rightarrow$
- Convergence: W^ν lop-converges ancillary-tight to W

MaxInf but not MinSup

- Variational Inequality: $-G(x) \in N_C(x)$: $F(x,y) = \langle -G(y), y-x\rangle$ on $C \times C$
- Nash Equilibrium, non-cooperative game: $\min r_i(x_i, x_{-i}),\ i \in \mathcal{I}$,
  $F(x,y) = \sum_{\mathcal{I}} \big(r_i(x_i, x_{-i}) - r_i(y_i, x_{-i})\big)$
- Complementarity: $0 \le x \perp H(x) \ge 0$, $F(x,y) = \langle H(x), y-x\rangle$ on $\mathbb{R}^n_+ \times \mathbb{R}^n_+$
- Fixed point: $S(x) \ni x$ (on C): $F(x,y) = \sup\big\{\langle y-x, z-x\rangle \,\big|\, z \in S(x) \subset C\big\}$
- MPEC: $\max g(x),\ x \in S(x)$: $F(x,y) = g(x) + \sup\big\{\langle y-x, z-x\rangle \,\big|\, z \in S(x)\big\}$
- "Classical" bifunctions: Lagrangian, augmented Lagrangian, Hamiltonian, optimality functions (Polak), ..
- New domains: PDE (Mountain Pass Theorem), minimal surfaces, ???

Convergence
finite-valued on product sets





Lopsided Convergence
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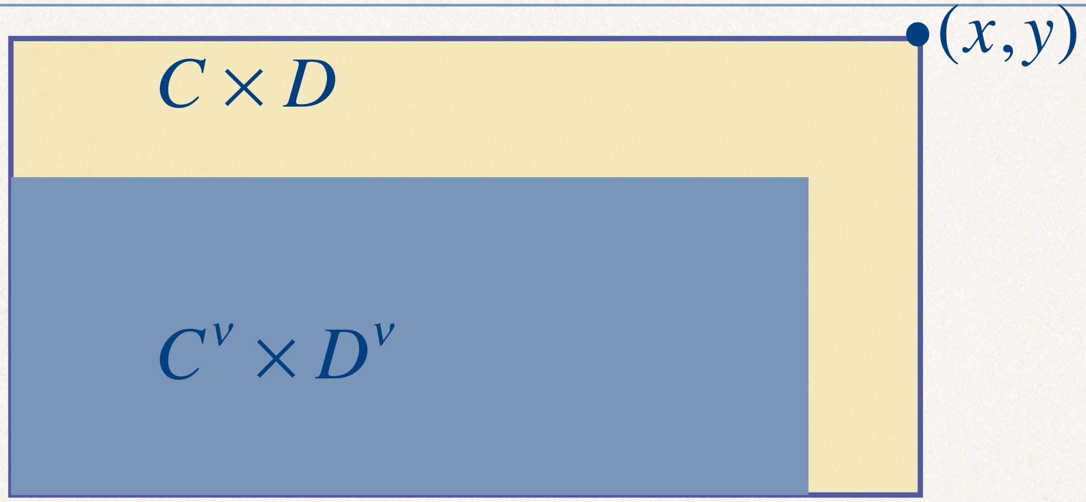

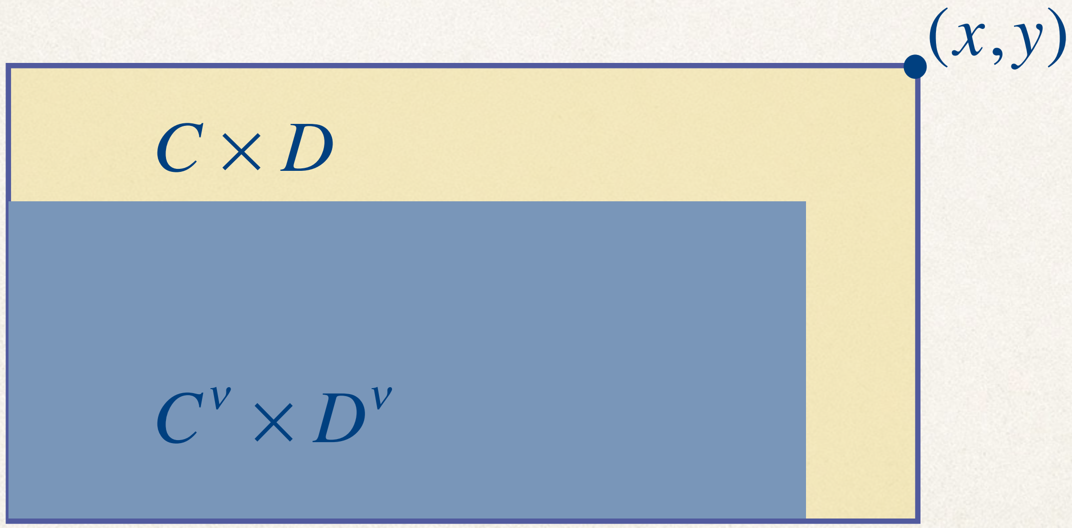

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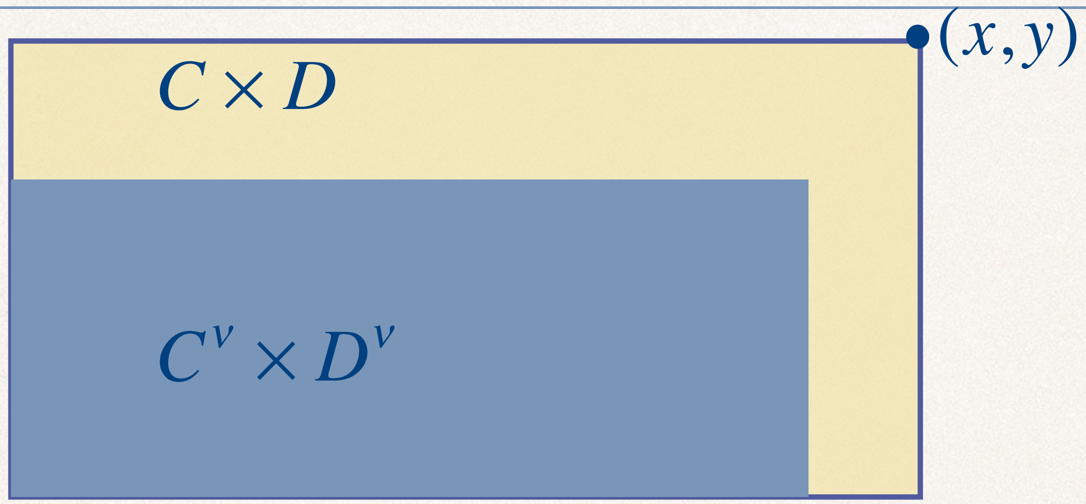
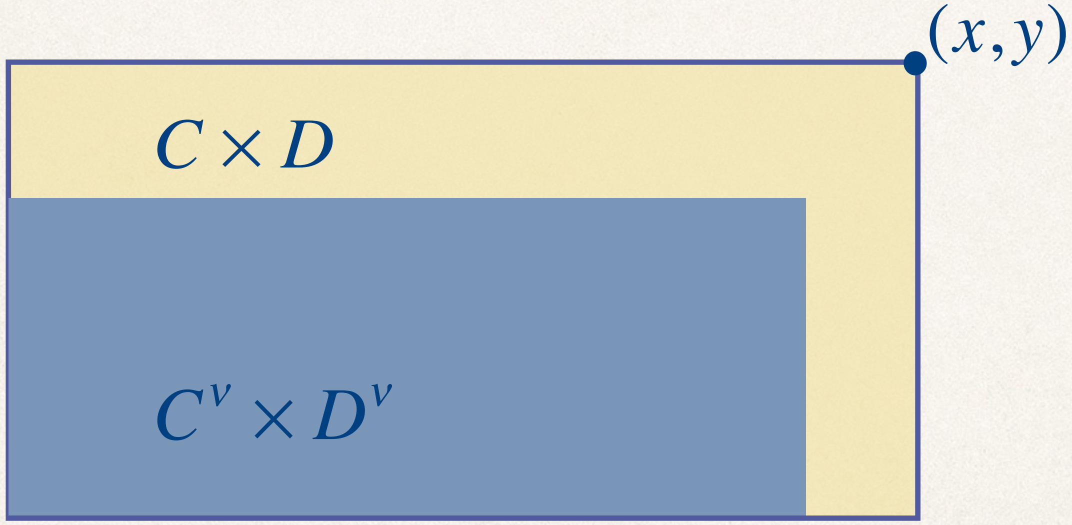


$$\forall y \in D, \forall x^{\nu} \in C^{\nu} \to x$$



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$\exists y^\nu \in D^\nu$

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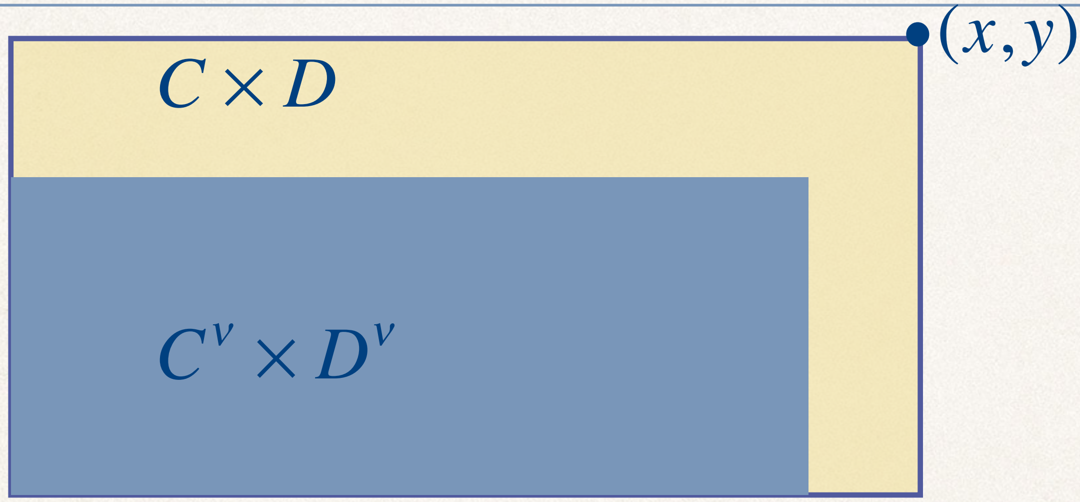

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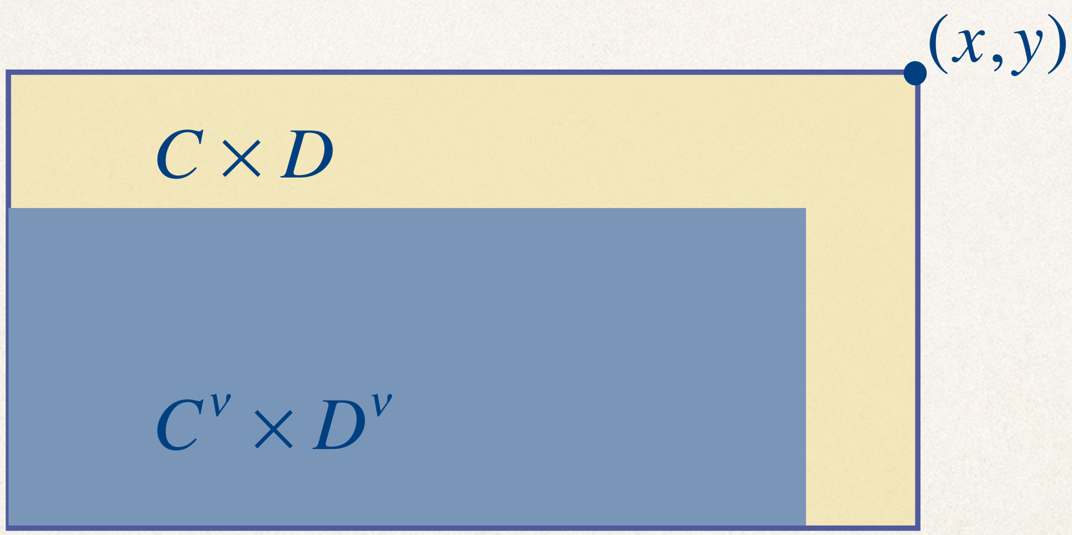

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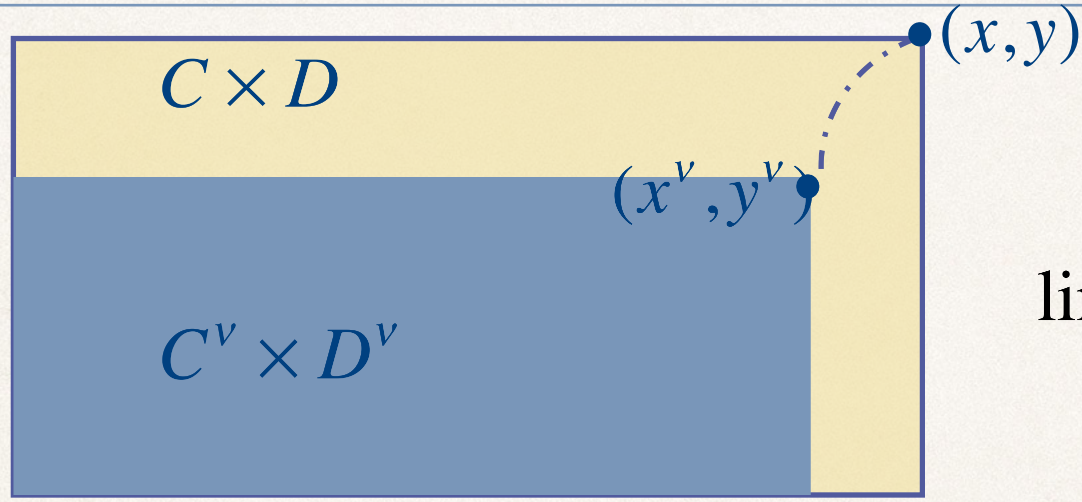

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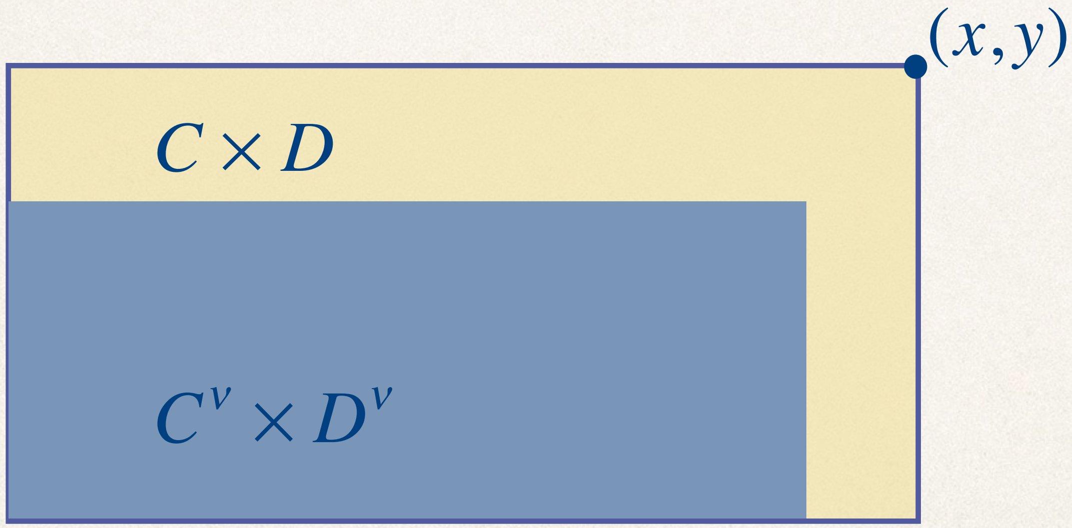

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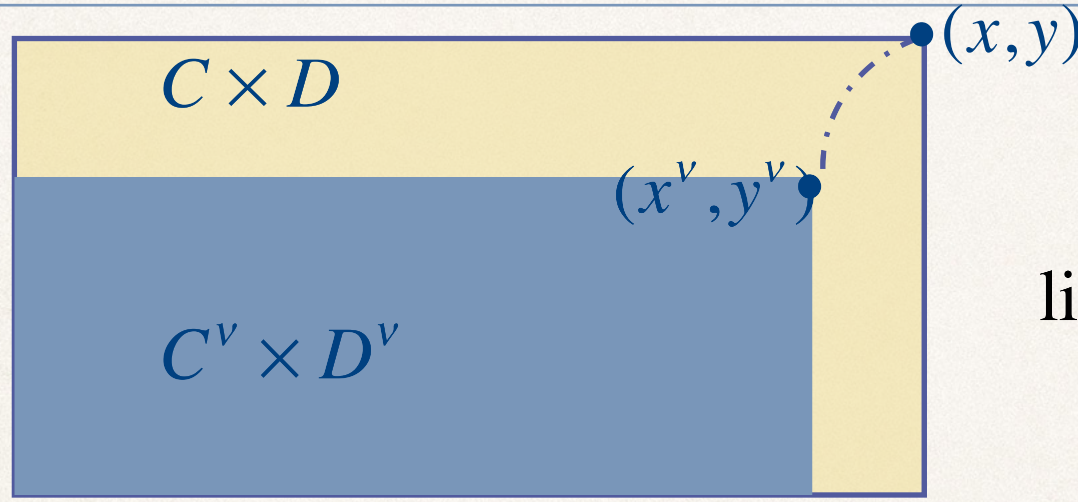

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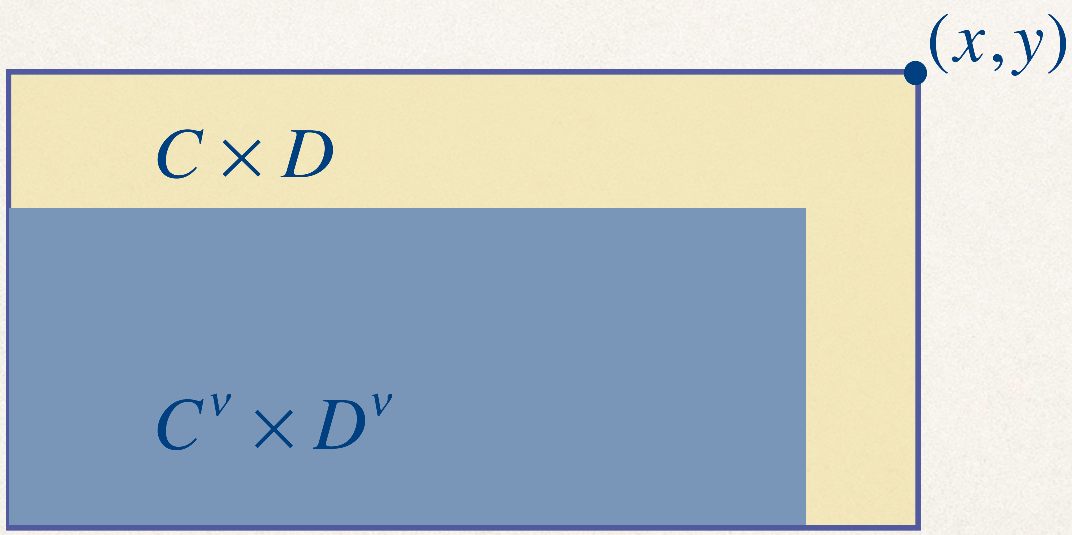

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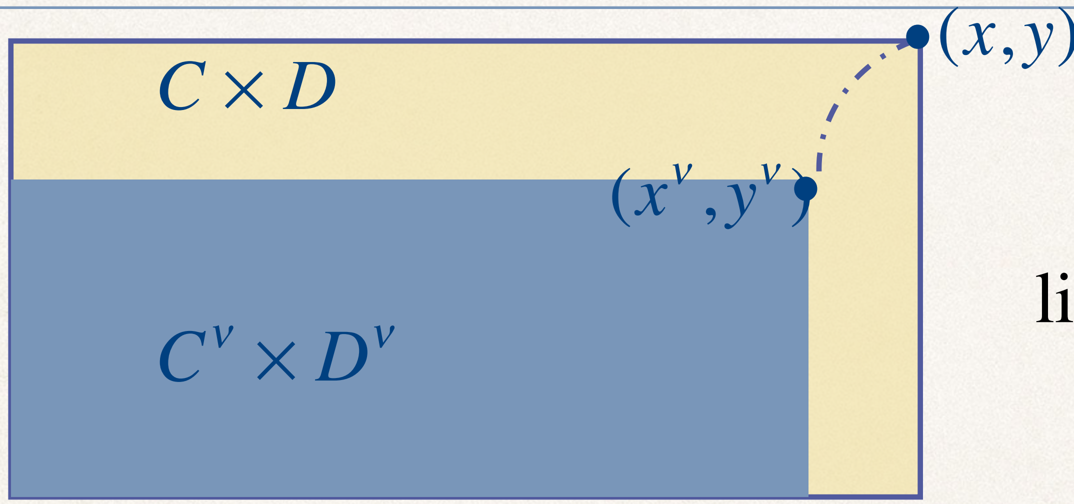

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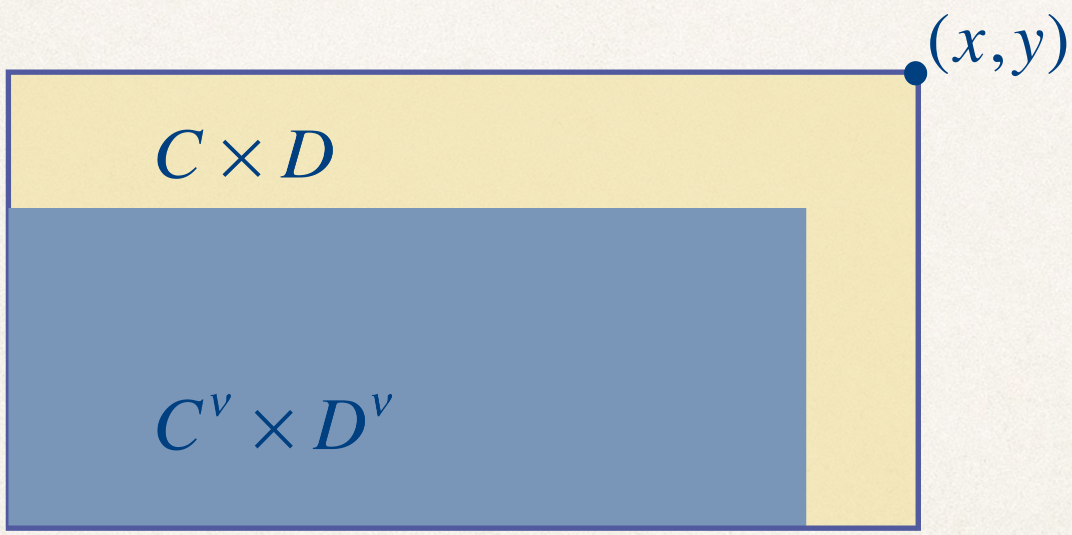

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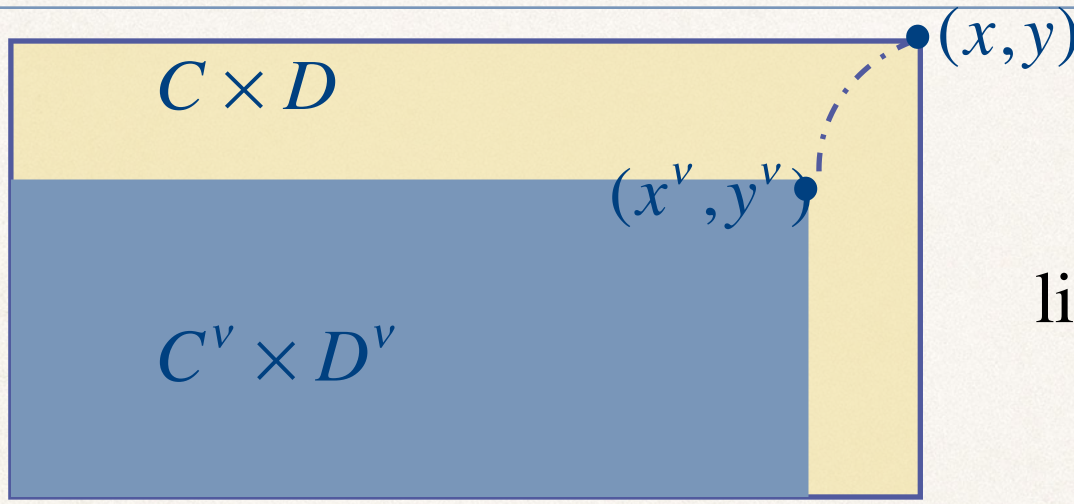

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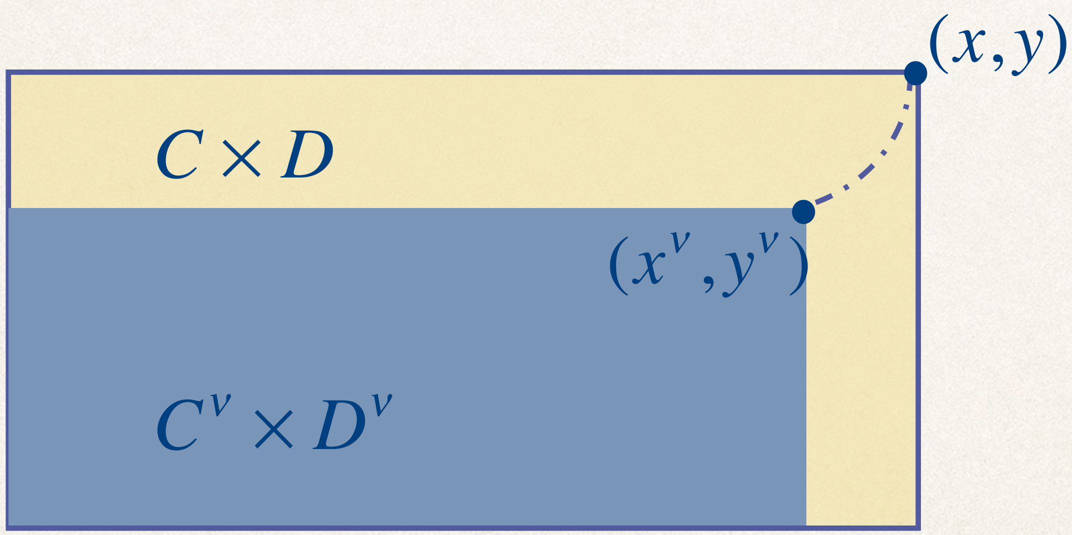

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$\limsup_\nu K^\nu(x^\nu, y^\nu) \leq K(x,y)$ when $y \in D$

$K^\nu(x^\nu, y^\nu) \to -\infty$ when $y \notin D$

Epi/Hypo-Convergence

finite-valued on product sets

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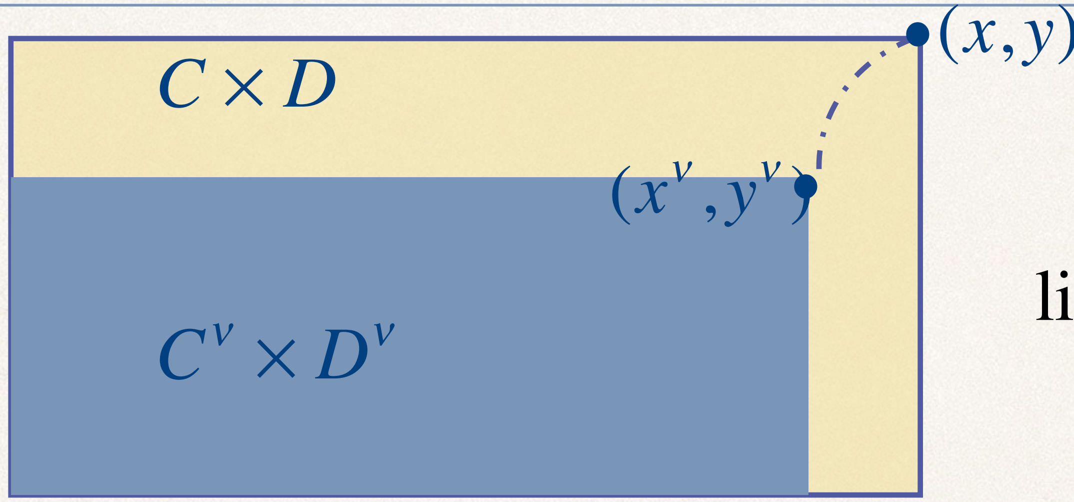

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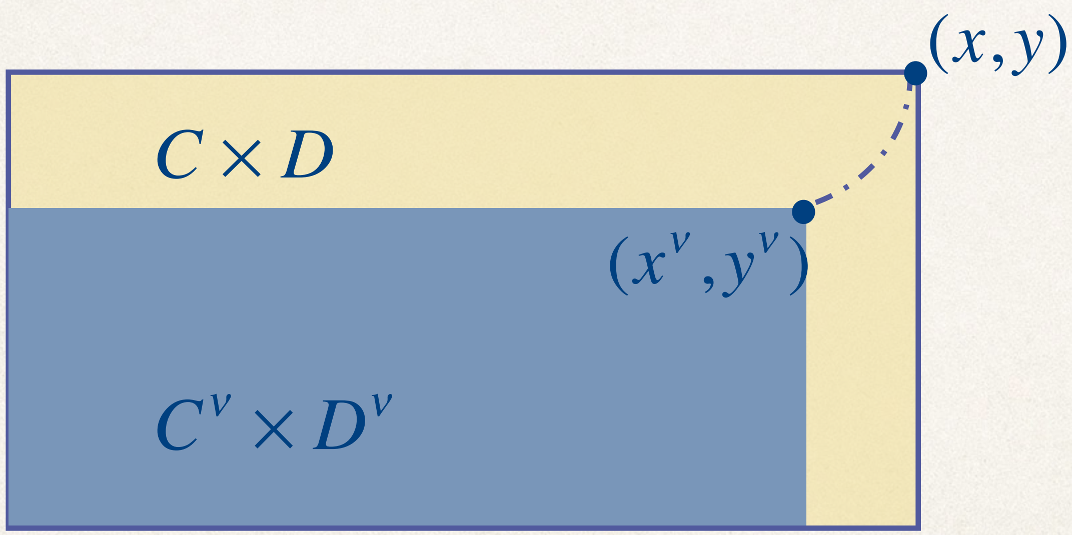

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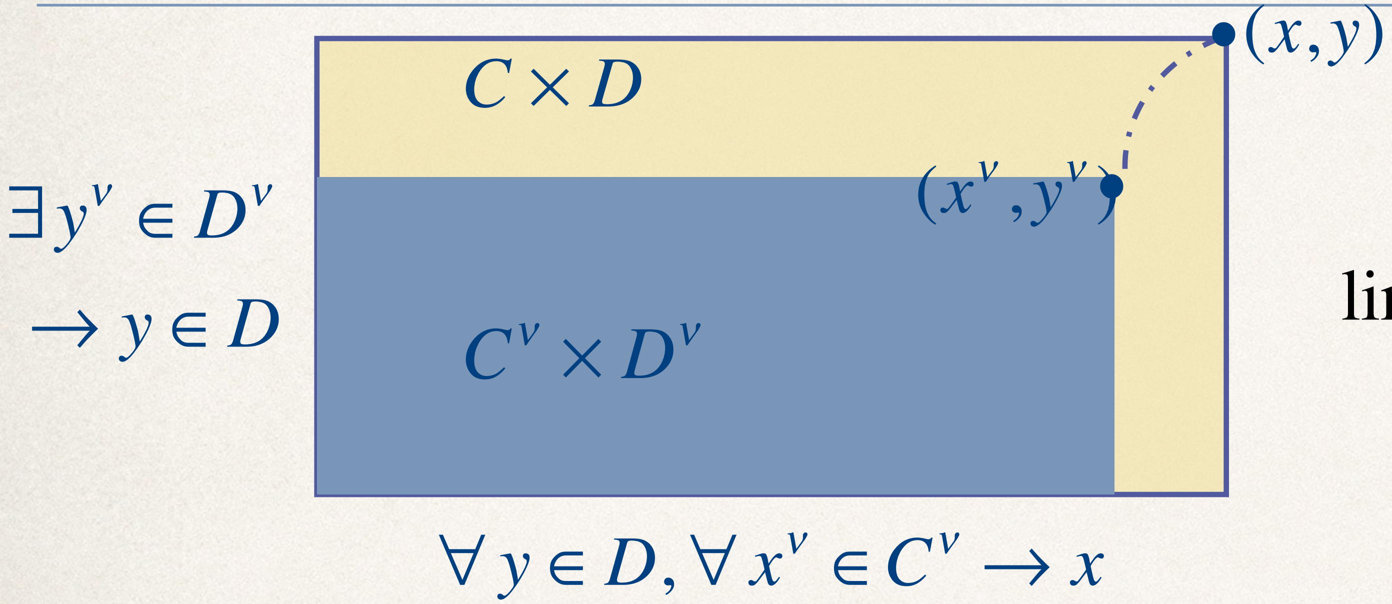

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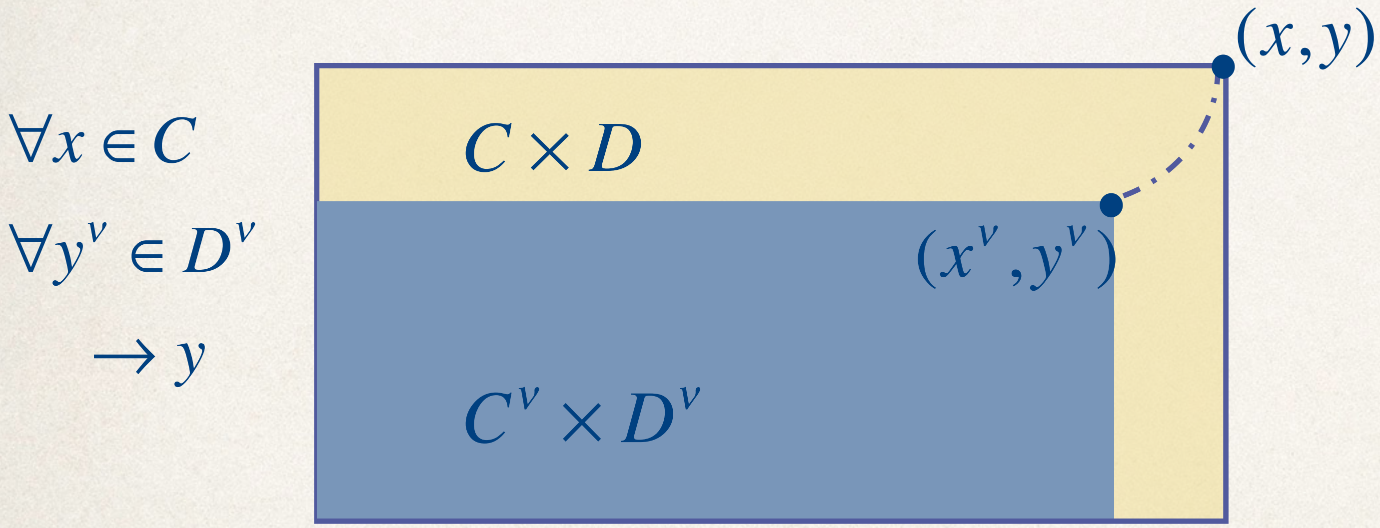

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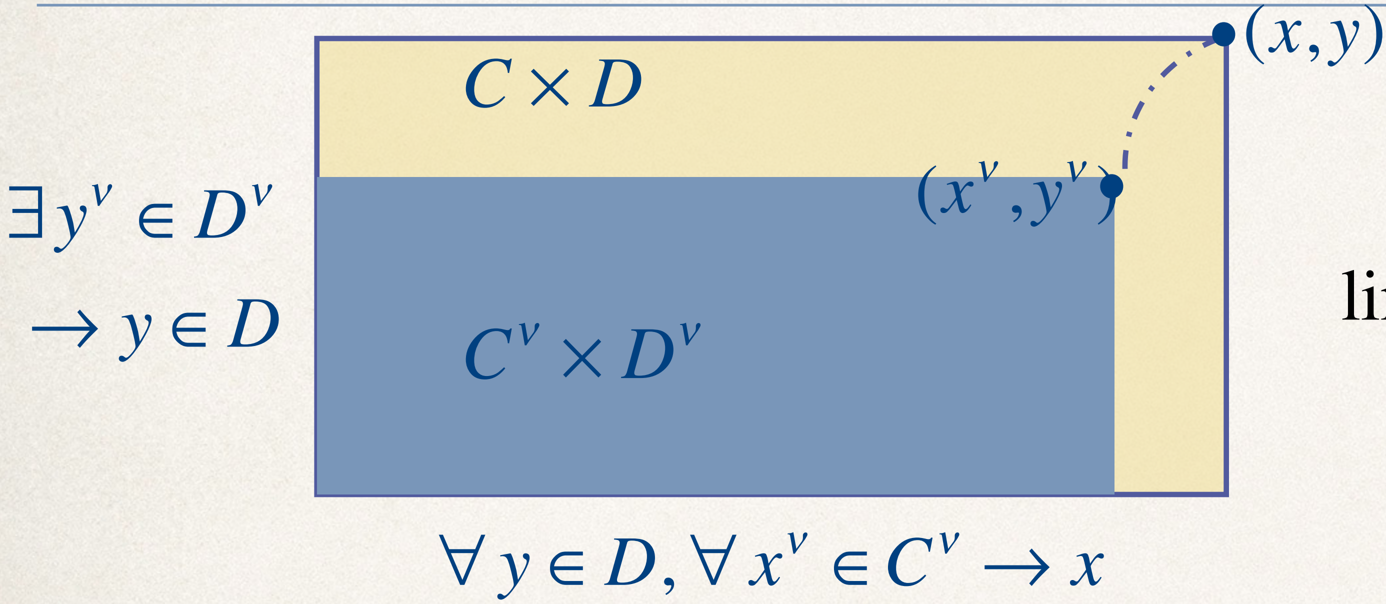

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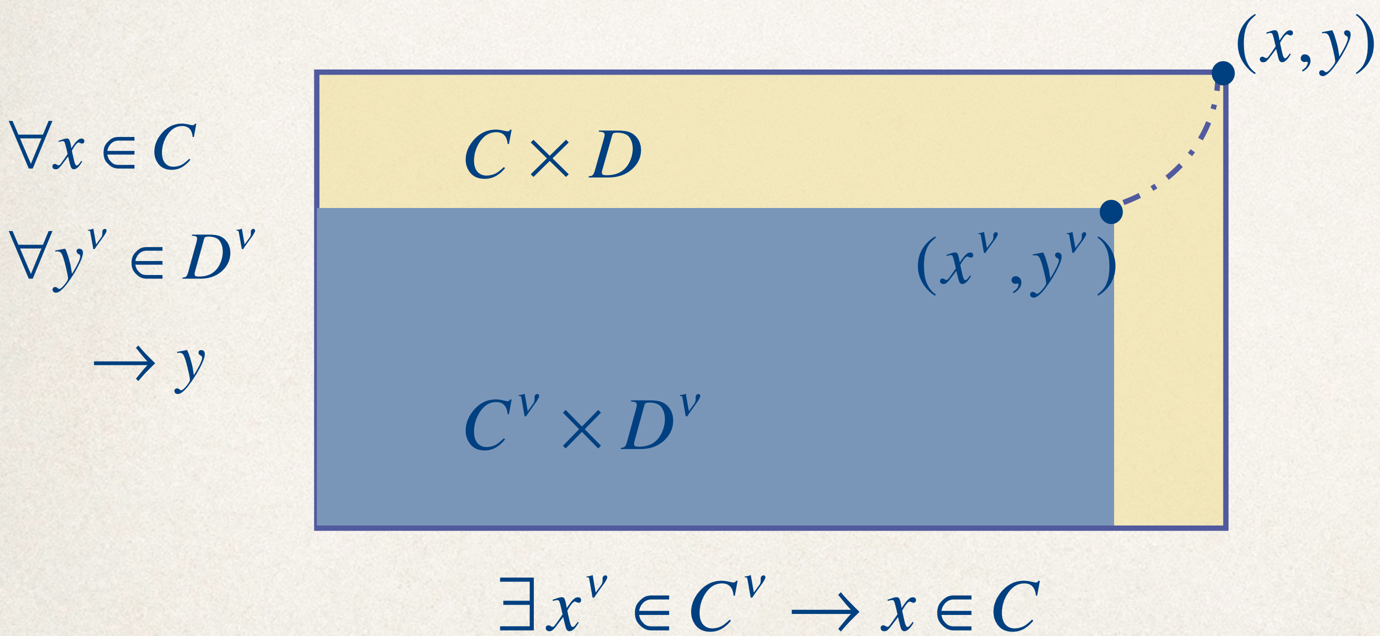

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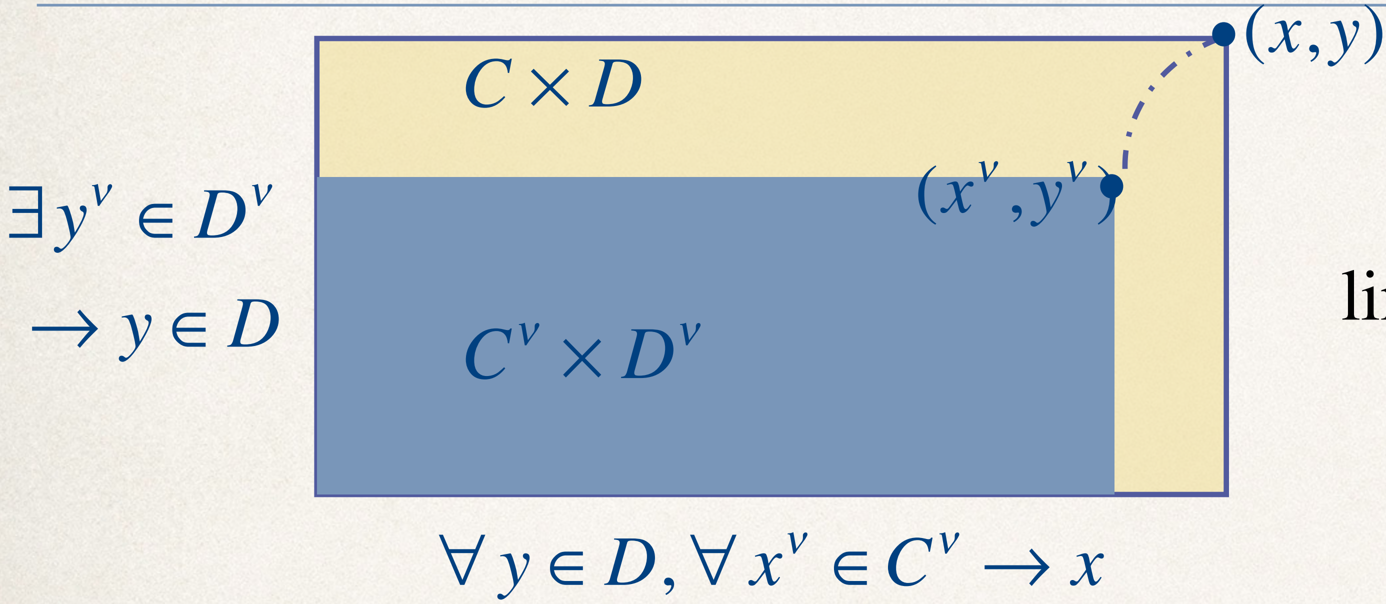

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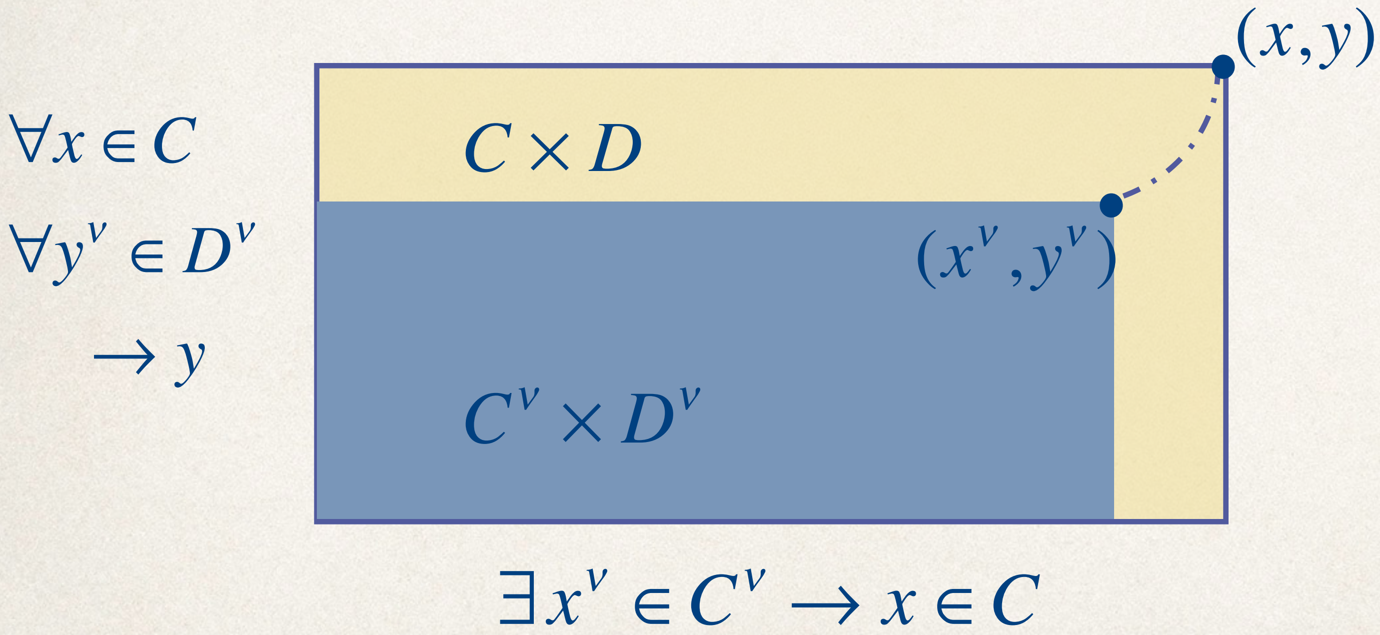

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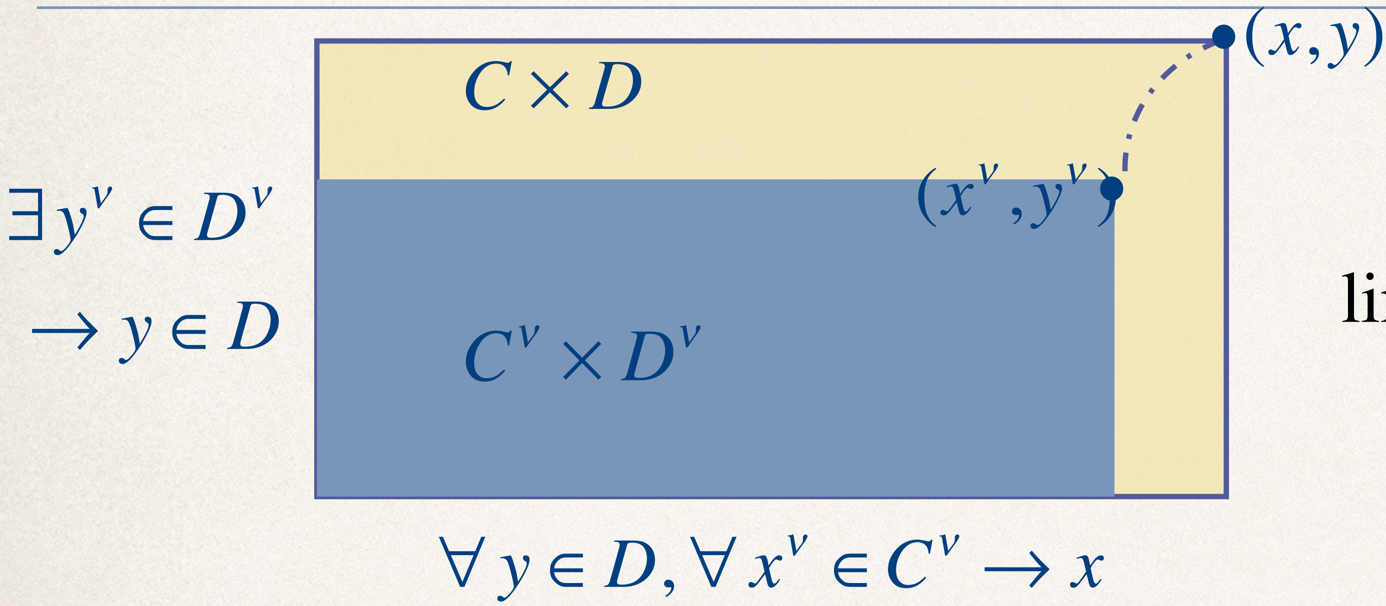

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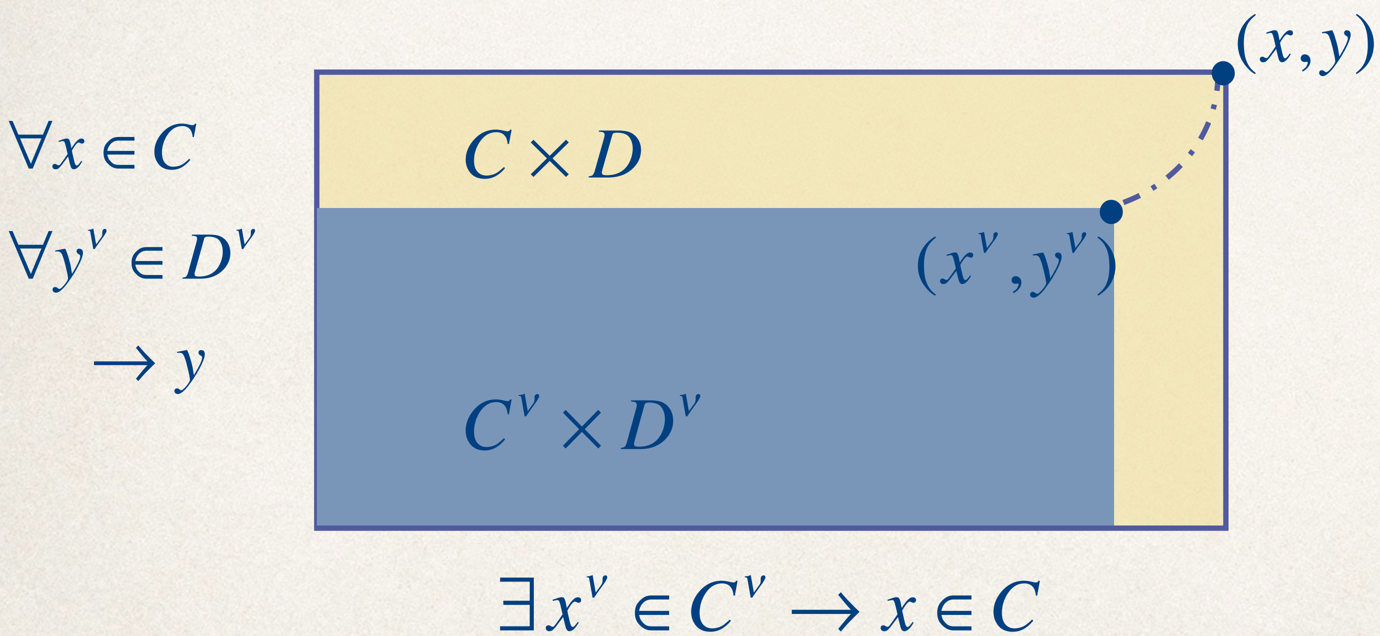

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Fenchel rehabilitation ?

Convergence solutions and of approximating solutions

$F^{\nu}_{C^{\nu}\times D^{\nu}} \to F_{C\times D}$ lop. ancillary-tightly (~ relaxed-compact in y)

(i) $x^{\nu} \in \varepsilon$-MnSp $F^{\nu}_{C^{\nu}\times D^{\nu}}$, $\bar{x}$ cluster point of $\{x^{\nu}\}_{\nu\in\mathbb{N}}$

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(iii) $\bar{x} \in$ MnSp $F_{C\times D} \Rightarrow \exists \varepsilon_{\nu} \searrow 0$ & $x^{\nu} \in \varepsilon_{\nu}$-MnSp $F^{\nu}_{C^{\nu}\times D^{\nu}}$

such that $x^{\nu} \to \bar{x}$,

Under tight-lop: convergence of the full ε_{ν}-maxinf sets

and convergence of values

more about Equivalence Classes

sup-projection: $h(x) = \sup_y F(x,y),\ h : C' \subset C \to \mathbb{R}$

$$F^\nu \xrightarrow{lop} F \quad \Longrightarrow \quad h^\nu \xrightarrow{e} h$$ epi-convergence

F, G in the same *equivalence class* if their sup-projections coincide.

? Induced metric: $d\!l_{bifcns}(F,G) = d\!l(h_F, h_G)$?

L'oeuf de Colombe

some connection(s) for saddle functions

L'oeuf de Colombe

MinSup-framework or MaxInf-framework

some connection(s) for saddle functions

Uniqueness Theorem: MaxInf-Framework

Limit functions $\qquad \nu$ filter: $\mathcal{H}$

e-limit: $l_e F^\nu = \sup_{U\in\mathcal{N}_\tau(x)} \inf_{V\in\mathcal{N}_\sigma(y)} \inf_{H\in\mathcal{H}} \sup_{\nu\in H} \sup_{v\in V\cap D^\nu} \inf_{u\in U\cap C^\nu} F^\nu(u,v)$

upper-lop: $lop_u\text{-}F^\nu = \sup_{U\in\mathcal{N}_\tau(x)} \inf_{V\in\mathcal{N}_\sigma(y)} \sup_{H\in\mathcal{H}} \inf_{\nu\in H} \sup_{v\in V\cap D^\nu} \inf_{u\in U\cap C^\nu} F^\nu(u,v)$

F is lop-limit if $l_e F^\nu \leq F$ and $\forall\, y \in D$,

$x \in C : lop_u\text{-}F^\nu(x,y) \geq F(x,y), \;\; x \notin C : lop_u\text{-}F^\nu(x,y) \geq \infty$

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UNIQUENESS *of lopsided limits in the MaxInf-framework*

Uniqueness Theorem: MaxInf-Framework

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UNIQUENESS *of lopsided limits in the MaxInf-framework*

applies also in the SupInf-framework (with a different lop-limit, as expected)

for Saddle Functions: the MaxInf- and SupInf-lop limits
bracket the epi/hypo-equivalent class


Terry!
to Minimal Surfaces